Ocean waves – wave pressure

The Danish crown prince ***weightless*** up in the air - (despite his mass of several kilograms).

Likewise will high splashing water from a standing wave by a vertical face breakwater lose weight.

So below a high wave crest, at still water level, will the water pressure be substantially lower than the mass weight of the above water (= hydrostatic pressure, the pressure traditionally used)?

Danish crown prince Frederik is here waved into the air by his father prince Henrik in Fredensborg garden.

Portrait of the Danish royal family from 1974, featuring glimpses from the Queen's private and official daily life. The program was made in collaboration with the BBC. Can be seen on youtube:

https://www.youtube.com/watch?v=dVsRXsAPoAE: Queen Margrethe of Denmark: A portrait (1974)

Improvement of the simple water wave formulas

The classical wave theory can easily be improved to give more convincing simple formulas for wave pressure and water particle velocities, especially of importance the pressure close to the surface. This is shown here below for the simple standing wave at a vertical face breakwater, and before that shown for the simple (1' order) deep water progressive wave, and also for the deep water (2' order) cnoidal wave.

For each wave: one page with simple formulas, including the proof of its correct wave theory (page 3, 4, 8).

Wave topping with vertical acceleration

When lifting a child in your arms and for fun waving it high up and down then it is easy to feel that its weight is felt different when the child is down and when it is high up. A 7 kg weight is felt maybe like 3 kg or less when high up. This is a result of Newton's 2' law for the equation of motion with the child's downward acceleration.

We have the same phenomena when the water waves move up and down by a vertical wall. This could be a vertical face concrete breakwater standing in 10 m deep water subjected to ocean waves moving 5 m up and down. So when the wave has crest then the water pressure on the wall will be less than the normal weight of water all the way from the water top surface and down, because of the water's downward acceleration. So the pressure is less than the hydrostatic pressure predicted by the classical wave theory.

Traditional classical wave theory, a mathematical potential theory

In the 19' century theories with the mathematical description of regular water waves were published. They were excellent theories, with necessary minor mathematical physical approximations, e.g. a so called 1' order theory for waves of small wave heights, with simple relevant formulas yielding recognizable practical results, widely used by engineers. This excellent 1' order theory shows that the increase in the water pressure on the wall, the wave pressure, when there is crest, decreases downwards according to the hyperbolic cosine function, cosh. This is in agreement with Newton's 2' law of motion because of the vertical acceleration when the water turns from moving upwards to moving downwards. But the cosh is used from the mean water level (MWL) and down in this potential wave theory, and not up all the way down from the surface where the acceleration is biggest. This wave theory is based on a mathematical potential theory not entirely operating directly with the physics of the water. The result for bigger waves deviates to some extent in some formulas relevant for the structural harbor engineer, e.g. wave pressure.

Direct physical wave theory– new simple easy proven formulas– deep water waves and standing waves

With an alternative mathematical description of the wave theory based directly on the physics of the water these simple 1' order formulas can easily be improved to be more engineer trustworthy for waves of relevant size in designing structures. The crest pressure at mean water level MWL is not hydrostatic, it is noticeable less. And the pressure at the trough surface is = 0, so the sucking pressure of the wave trough is bigger than according to the potential theory. The improved formulas of 1' order approximation with theoretically verifications are shown in the following 2 short examples of wave descriptions: the progressive deep water wave (from the next page), and the general standing wave (from page 6).

Deep water 1’ order sinusoidal wave and 2’ order cnoidal wave

The smooth regular ocean waves we see in straight rows coming towards us rolling our boat up and down they got in 1845 Airy to publish his famous much applied classical theory for simple periodic gravitational surface water waves: the sinusoidal wave. This theory gave us the natural connection between the wave period T and wave length L and thereby the wave celerity c. The wave surface elevation η is described by a sine function (or cosine) with a vertical wave height H from trough to crest. Formulas for wave pressure and horizontal and vertical particle velocity at any point in the water were given (of 1’ order approximation).

The formula for the wave surface elevation η for a progressive wave is:

$\eta = H/2 \cos \theta_1$

$\theta_1 = k(x - ct)$, $k = 2\pi/L$, t is the time, x and z are the horizontal and vertical coordinates, z = 0 at mean water level, MWL; wave crest $\eta_c = H/2$ at $\theta_1 = 0$, wave trough $\eta_t = -H/2$ at $\theta_1 = \pi$, and in between $\eta = 0$ at $\theta_1 = \pi/2$.

But for big waves the crest will be taller than H/2 and more narrow than π, and the wave trough less deep than z = -H/2 and longer than π. This wave form is particular apparent for shallow waters, so in 1895 a special wave theory was published for shallow water waves called cnoidal waves. To describe these waves using the cos θ_1 function we wish the θ_1-axis to “shrink” by the crest and to “expand” by the trough “so that π/2 moves closer to $\theta_1 = 0$”. For this purpose we can use a Jacobi mathematical elliptic function. And this function can be used not only for traditional shallow water waves, but all the way out to deep water waves.

Progressive deep water waves

With $\theta_1 = 2\theta$ the sinusoidal wave surface elevation can be written as

$\eta = H \cos^2\theta - H/2$, where H/2 = trough depth

In the same way we write the cnoidal wave with Jacobi’s elliptic cosine function:

$\eta = H\, \mathrm{cn}^2\theta$ - trough depth

where an elliptic parameter m determines “how high and narrow” the wave crest is, and determines the elliptic integrals of first and second kind: K and E (given by table or computer). We will here show that the $\mathrm{cn}^2\theta$ we know from shallow water waves can also be used for even 2’ order deep water waves.

The Jacobi elliptic parameter m is in my wave description found to be the rather simple expression:

$m K^2 = \pi^3 H/L$ for deep water. H/L is wave steepness, and H/L is max 14% in nature.

(For any arbitrary water depth D we have: $m K^2 = \pi^3 H/L \coth^3 kD$, all the way to including the solitary wave in shallow waters, with the same formulas for all regular progressive waves in a wave description of 2’ order, and with better fulfillment of boundary conditions than in the traditional wave theories).

The cnoi deep water wave profile here differs very little from the traditional 2’ order sinusoidal wave, but the formulas here comply better with boundary conditions, so they are more relevant for practical use.

Formulas for the 1’ order deep water wave

$\eta = H/2 \cos \theta_1$

$c^2 = g/k$, so: $L = c\,T$; $k = 2\pi/L$

$u = c\,\eta\,k\,e^{k(z-\eta)}$ for the horizontal particle velocity

$w = w_s\,e^{k(z-\eta)}$, where the vertical particle velocity at the surface $z = \eta$ is $w_s = \partial\eta/\partial t = -c\,\partial\eta/\partial x$

$p/\gamma + z = \eta\,e^{k(z-\eta)}$ is the water pressure; below MWL and trough surface: the wave pressure: $p^+/\gamma = \eta\,e^{k(z-\eta)}$

$e^{k(z-\eta)}$ pressure reduction downwards in our wave theory here starts from the wave surface, and not from MWL as in the classical potential theory. The formula here gives for the wave crest the expected acceleration reduced wave pressure at MWL ($z = 0$), and gives pressure $p = 0$ at the trough surface, also different for the potential theory. (Example: a North Sea drilling rig in deep water with L = 200 m wave with (measured) wave crest height =14 m gets at MWL ($z = 0$) a wave pressure = 9 m water pressure in the theory here, and 14 m water pressure by the Airy formula. So this simple formula for p/γ , fulfilling precisely the surface condition, gives a not insignificant reduction in calculation of pressure.)

To prove the validity of the wave theory the formulas here will then for an ideal fluid be shown to fulfill the governing equations: **conservation of mass** and the **dynamic equations** to the 1’ order approximation:

$\partial u/\partial x = c\,\partial\eta/\partial x\,k\,e^{k(z-\eta} - c\,\eta\,\partial\eta/\partial x\,k^2 e^{k(z-\eta)}$; the last term: $\eta\,\partial\eta/\partial x$ is of 2’ order in H/L, so: negligible here.

$\partial w/\partial z = -\,c\,\partial\eta/\partial x\,k\,e^{k(z-\eta)}$

Conservation of mass: $\partial u/\partial x + \partial w/\partial z = 0$ is then seen fulfilled for a 1’ order theory.

Horizontal acceleration: $G_x = du/dt = \partial u/\partial t + u\,\partial u/\partial x + w\,\partial u/\partial z = c\,\partial\eta/\partial t\,k\,e^{k(z-\eta)}$ + 2’ order terms

For a progressive wave we have: $\partial\eta/\partial t = -\,c\,\partial\eta/\partial x$

$\partial p/\partial x = \gamma\,\partial\eta/\partial x\,e^{k(z-\eta)}$ (+ 2’ order terms) is seen to fulfill:

Horizontal dynamic equation: $\partial p/\partial x = -\rho\,G_x = \rho\,c^2\,\partial\eta/\partial x\,k\,e^{k(z-\eta)} = \rho\,g/k\,\partial\eta/\partial x\,k\,e^{k(z-\eta)} = \gamma\,\partial\eta/\partial x\,e^{k(z-\eta)}$

Vertical acceleration: $G_z = dw/dt = \partial w/\partial t + u\,\partial w/\partial x + w\,\partial w/\partial z = -\,c\,\partial^2\eta\,/\partial x\,\partial t\,k\,e^{k(z-\eta)}$ + 2’ order terms

$\partial p/\partial z + \gamma = \gamma\,\eta\,k\,e^{k(z-\eta)}$ is seen(using 1’order $\partial^2\eta/\partial x^2 = -k^2\,\eta$) to fulfill:

Vertical dynamic equation: $\partial p/\partial z + \gamma = -\rho\,G_z = \rho\,c\,\partial^2\eta/\partial x\partial t\,e^{k(z-\eta)} = -\rho\,c^2\,\partial^2\eta/\partial x^2\,e^{k(z-\eta} = \gamma\,\eta\,k\,e^{k(z-\eta)}$

It is seen that we did not need to assume irrotational motion. (Wanted rotation can be investigated afterwards). The wave theory here is not meant as just another way to develop the already known 1’ order wave theory, but to show that the better wave formulas here fulfill the wave equation correctly. A 1’ order theory should include those 2’ order terms that obvious improve the result as seen in the formulas here.

γ = weight of water 10 kN/m^3, ρ = unit mass, g = acceleration of gravity = 9,81 m/sec$^2 \approx$ 10 m/sec^2
1 m water pressure = 10 kN/m^2

Formulas for the 2’ order cnoi deep water wave

$\eta = H\, cn^2\theta + \eta_t$; (η_t is the negative trough depth, $z = \eta_t$)

$\theta = 2K/L\,(x - c\,t)$

$\eta_t = H/m\,(1 - m - E/K)$; so: $\eta_c = H/m\,(1 - E/K)$; η_c is wave crest height, η_t is the (negative) trough depth

$m\,K^2 = \pi^3\, H/L$ for deep water

$c^2 = g/k$, so: $L = c\,T$; $k = 2\pi/L$

$u = c\,\eta\,k\,e^{k(z-\eta)}$ for the horizontal particle velocity

$w = c\,\partial\eta/\partial x\,(-1 + \eta\,k)\,e^{k(z-\eta)}$ for the vertical particle velocity

u and w here comply fully with the kinematic surface boundary condition (not just approximately as for the potential theory).

$p/\gamma + z = \eta\,e^{k(z-\eta)} + \pi/4\,H^2/L\,e^{k(z-\eta)}\,(1 - e^{k(z-\eta)})$; as the proposed approximate water pressure, (see below),

water pressure p fulfills $p = 0$ at the surface: $z = \eta$, and wave pressure $p+ = p - z = 0$ at big depth: $z \rightarrow -\infty$

$\partial u/\partial x = c\,k\,\partial\eta/\partial x\,(1 - \eta\,k)\,e^{k(z-\eta)}$

$\partial w/\partial z = c\,\partial\eta/\partial x\,(-1 + \eta\,k)\,k\,e^{k(z-\eta)}$

So the **conservation of mass**: $\partial u/\partial x + \partial w/\partial z = 0$ is seen fulfilled here in this 2’ order theory

Horizontal accel $G_x = du/dt = \partial u/\partial t + u\,\partial u/\partial x + w\,\partial u/\partial z = c\,\partial\eta/\partial t\,k\,e^{k(z-\eta)}\,(1 - \eta\,k) = -c^2 k\,\partial\eta/\partial x\,e^{k(z-\eta)}\,(1 - \eta\,k)$

(the convective terms are: $u\,\partial u/\partial x + w\,\partial u/\partial z = 0$).

$\partial p/\partial x = \gamma\,\partial\eta/\partial x\,e^{k(z-\eta)}\,(1 - \eta\,k)$ (+ 3’ order terms) is seen to fulfill:

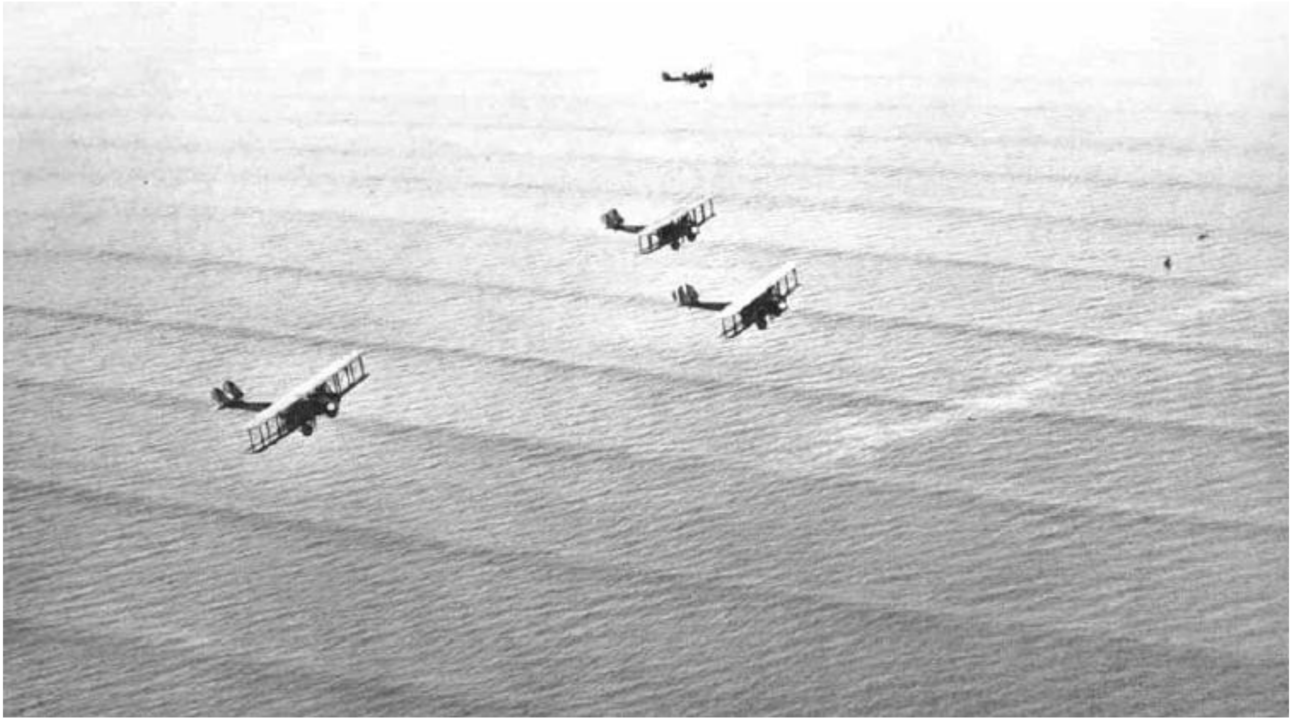
Horizontal dynamic equation: $\partial p/\partial x = -\rho\,G_x = \rho\,c^2\,\partial\eta/\partial x\,k\,e^{k(z-\eta)}\,(1 - \eta\,k) = \gamma\,\partial\eta/\partial x\,e^{k(z-\eta)}\,(1 - \eta\,k)$

The vertical dynamic equation is too laborious for this page.

In making the wave equation to find the cnoi wave solution here, including determine mK^2 , the vertical dynamic equation gives an expression for the pressure to be used in the horizontal dynamic equation, as described in the reference to my book below.

From the internet:
https://en.wikipedia.org/wiki/Cnoidal_wave#/media/File:Periodic_waves_in_shallow_water.jpg

A cnoidal wave, characterised by sharper crests and flatter troughs than in a sine wave. For the shown case, the elliptic parameter is $m = 0.9$.



US Army bombers flying over near-periodic swell in shallow water, close to the Panama coast (1933). The sharp crests and very flat troughs are characteristic for cnoidal waves.
(I have seen a likewise group of cnoidal waves coming towards a beach of constant depth at Køge Bugt in Denmark a nice summer day. I wish I then had the instruments to measure: T, c, L, η, u, w, p).

For practical use of also the deep water cnoidal wave I would for pressure just use the simple reduced formula: $p/\gamma + z = \eta\, e^{k(z-\eta)}$ with η from the cnoi wave. This pressure formula gives an expected pressure reduction also in the wave above mean water level MWL, (where the potential theory gives hydrostatic pressure). The precise 2' order pressure formula (to be used in developing the wave theory) is (unabbreviated in 2' order terms):

$$p/\gamma + z = \eta + 1/k\{[1/k\, \partial^2\eta/\partial x^2 - 2\, (\partial\eta/\partial x)^2 - \eta\, \partial^2\eta/\partial x^2][1- e^{k(z-\eta)}] + ½[(\partial\eta/\partial x)^2 - \eta\, \partial^2\eta/\partial x^2][1- e^{2k(z-\eta)}]\}$$

Here follows a few math "cnoi formulas":

$\eta = H\, cn^2\theta + \eta_t$ (here η_t is negative)

η_t is found by integrating η over a wave length L to give = 0: $\int_0^{x=L} \eta\, dx = 0$

$$\partial\eta/\partial x = -4K\, H/L\, \sqrt{[(cn^2\theta\, (1- cn^2\theta)(1 - m + m\, cn^2\theta)]}$$

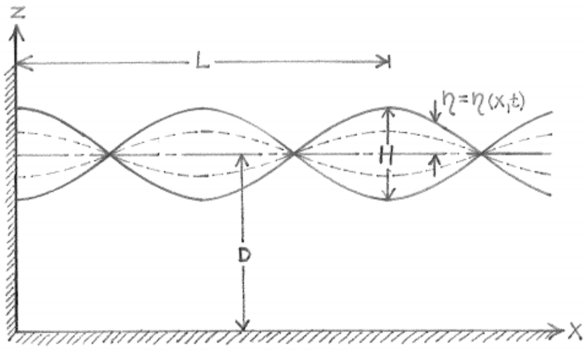
$$\partial^2\eta/\partial x^2 = -8K^2\, H/L^2\, (-1 + m - 2\, (2m - 1)\, cn^2\theta + 3m\, cn^4\theta))$$

$$\partial^3\eta/\partial x^3 = 64K^3\, H/L^3\, (1 - 2m + 3m\, cn^2\theta)\, \sqrt{[(cn^2\theta\, (1 - cn^2\theta)(1 - m + m\, cn^2\theta)]}$$

Further description of this wave theory and cnoi waves on arbitrary depth, and other regular waves, see:
http://lavigne.dk/waves/wavese.htm or http://www.mejlhede.dk/

Wave pressure in a standing wave

When moderate regular ocean waves travel towards a vertical face breakwater we observe that the waves become standing waves: the water at the vertical wall oscillates regular up to a crest and down to a trough.



The height from the trough to the crest is the wave height H. The time from one crest at the wall to the next crest is the wave period T. At a distance from the wall we call the wave length L the water will oscillate with a crest at the same time as at the wall, while at the distance L/2 from the wall the water will oscillate opposite also with the wave height H. (L depends on the wave period T and the mean water depth, as decided by the wave theory).

Water pressure example

What will the water pressure on the wall be when there is a wave crest and when there is a wave trough? When there are no waves the water pressure 1 meter below the surface is = 1 meter water pressure (= 10 kN/m^2). There is hydrostatic water pressure from the surface and down to the bottom, so that with 10 m water depth the pressure at the bottom is 10 m water pressure.
If the water at the wall has risen 5 m because of a high tide lasting for hours then this calm water will have a water pressure at 1 m depth of 1 m (= 1000 kg × acceleration of gravity 9,81 $m/sek^2 \simeq 10\ kN/m^2$ pressure), and at the bottom 15 m water pressure.
If the water instead because of T = 6 seconds storm waves from the ocean has risen to a 5 m crest in a short second with a downward turning acceleration of 5 m/sec^2 reducing the acceleration of gravity then the water pressure at 1 m depth is only ½ m = 5 kN/m^2. This follows of Newton's 2' law (momentum). And this is shown in our wave theory here.
With this moderate wave giving a crest at the wall of 5 m above the D = 10 m mean water depth then the pressure at the bottom will not be 10 + 5 = 15 m hydrostatic water pressure. The water by the wall is turning from upward movement to downward movement. So the water has a downward vertical acceleration, giving by Newton's 2' law that the pressure at the bottom will be less than 15 m water pressure. This less bottom pressure is also a result of the classical wave theories by Airy as well as Stokes from the 1800-s.

Considering the water pressure 1 m below the surface of the wave crest we found that it would be less than 1 m, and for the waves of design interest: the high waves, the pressure will be even less than ½ m, and this cannot be seen in the classical wave theories. So we want a better expression for the wave pressure, a formula that fulfills the surface conditions for pressure and acceleration and Newton's 2' law.

A different wave theory

When designing a vertical face breakwater we want to know what wave pressure it will get. And more: we would like to know the wave pressure and water velocities everywhere in the water, from the surface to the bottom and all over the wave length. A different wave theory based on necessary practical approximations in its theoretical development will be given with formulas here.

Approximations

For any flow of water we have the equation of continuity (conservation of mass), and we have the dynamic equation Newton's 2' law, to be used horizontally and vertically. But to get our wave theory we have to make some assumptions and some approximations:
At the sea bottom the water will move somewhat back and forth, with a little friction, so we may see some bottom sand moving a little. In our theory we neglect that friction, and also friction at the wall, and internal friction. We have an ideal fluid. (When our wave theory gives the horizontal water velocity at the bottom we have the possibility to maybe make moderations as practical engineers).

In our simple so called 1' order wave theory we consider the wave height H as small compared to the wave length L. So terms of higher order in the wave steepness H/L can be neglected.
When a 1' order theory is developed we can develop a 2' order theory by using the formulas of the 1' order theory in the 2' order terms.

At the wall the water moves vertically up and down, which according to the 1' order theory follows the sine (sinus) or cosine as a function of time t. By this we can calculate the vertical velocity and acceleration of the surface water. At the bottom the vertical velocity and acceleration is = 0. Then we can propose and try to prove in our wave theory if we can use that the acceleration from the surface and decreasing down to the bottom can be distributed as hyperbolic sine (sinh), that is all the way from the surface of the wave crest with a possible big negative acceleration and down to the bottom (and not just from the mean water level MWL and down). This will according to Newton's 2' law give that the wave pressure on the wall will have a cosh distribution all the way from the crest surface, giving e.g. that the wave pressure at the mean water level is less than hydrostatic pressure.

It is then to be proved that using this sinh vertical distribution and the equation of continuity and Newton's 2' law of momentum we will get the wave equation for a regular standing wave.

The mathematical development of the wave theory is given on: http://lavigne.dk/waves/Ch5.pdf
This is chapter V in my book on more new wave theories: "Regular Waves, 1977":
http://lavigne.dk/waves/wavesd.htm, or use my home page: www.mejlhede.dk

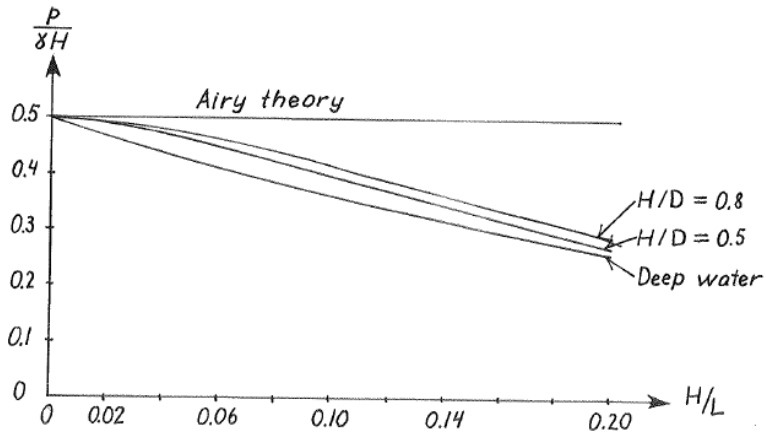


Figure 2: Wave pressure on a vertical wall at mean water level (MWL) below the wave crest of a regular wave, according to the 1' order theory. With a wave height of e.g. H = 1 meter the crest height is = 0.5 meter, so this gives a wave pressure of 0.5 m water (= 5 kN/m^2) according to the Airy theory.
Airy's classical theory does not include the pressure reducing effect of the "turning" acceleration of the above crest, which is included in my wave theory, and this is better according to experiments. Formula on the next page, illustrated in this figure for different wave steepness H/L and different mean depth D.

Formulas for the simple 1' order standing wave

In a coordinate system x,z with 0,0 at the foot point of the wall the equation for the surface profile measured from the mean water level MWL at z = D for the regular 1' order standing wave is:

$\eta = H/2 \cos(\omega t) \cos(kx)$

For a water particle with the coordinates x,z and time t we get the following formulas for water pressure p, and vertical velocity w, and vertical acceleration G_z , to be used anywhere in the water:

$p/\gamma = y - z + (G_s/g) \times (\cosh(Ry) - \cosh(Rz)) / (R \sinh(Ry))$ abbreviated to $p/\gamma = D - z + \eta \cosh(Rz) / \cosh(Ry)$
for water pressure anywhere, and for wave pressure p^+/γ above MWL

From this p/γ we get the wave pressure p^+/γ by subtracting the hydrostatic pressure from MWL: $p/\gamma = D - z$
(This hydrostatic pressure is the same as the water pressure on the calm harbor side of the breakwater).

Above the trough from surface to MWL we have negative hydrostatic pressure: $p^+/\gamma = p/\gamma = z - D$

Below the surface of trough and below MWL ($z \lesssim D$) for crest we get the wave pressure:

$p^+/\gamma = \eta + (G_s/g) \times (\cosh(Ry) - \cosh(Rz)) / (R \sinh(Ry))$; abbreviated (1' order) to $p^+/\gamma = \eta \cosh(Rz) /\cosh(Ry)$

$w = \partial\eta/\partial t \times \sinh(Rz) / \sinh(Ry)$

$G_z = \partial^2\eta/\partial t^2 \times \sinh(Rz) / \sinh(Ry)$; at the surface z= y: $G_s = \partial^2\eta/\partial t^2$

$(L/T)^2 = g/k \times \tanh(kD)$

$q = H/2 \times L/T \times \sin(\omega t) \sin(kx)$

$u = q \times R \times \cosh(Rz) / \sinh(Ry)$

$y = D + \eta$, where D = mean water depth, so y = actual water depth.

For the regular 1' order wave we have: $\partial^2\eta/\partial t^2 = - \omega^2 \times \eta$, $R = k = 2\pi/L$, $\omega = 2\pi/T$
γ = weight of water 10 kN/m^3, g = acceleration of gravity = 9,81 m/sec^2 $\approx$ 10 m/sec^2

Formulas for p^+, w, G, and u are of 1' order approximations, and in developing the wave theory the distribution of one of them is estimated and assumed, like sinh for the vertical acceleration, and then the theory gives the other formulas. (Or we can instead assume u to be cosh distributed).

For the surface $z = y$ ($= D + \eta$) we have:
pressure p = 0, vertical velocity $w_s = \partial\eta/\partial t$, vertical acceleration $G_{z=s} = G_s = \partial^2\eta/\partial t^2$
Wave pressure at the surface: $p^+/\gamma = 0$ above MWL, and $p^+/\gamma = \eta$ (negative) at wave trough surface.

We get the simple traditional classic Airy formula for wave pressure by substituting y = D in the above formula, as η in a 1' order theory is considered small and neglected:
$p^+/\gamma = \eta \times \cosh(kz) /\cosh(kD)$, but for z > D (above MWL) hydrostatic pressure is used.
This gives a bigger wave pressure than in my experiments and by my formula.
At the surface of the wave trough the Airy formula does not give the water pressure p = 0.

Test to prove the formulas for the standing wave of 1'order approximation

$\eta = H/2 \cos(\omega t) \cos(kx)$

$k = 2\pi/L$, $\omega = 2\pi/T$; $y = D + \eta$, where D = mean water depth, so y = actual water depth

For the regular 1' order wave we have: $\partial^2\eta/\partial t^2 = - \omega^2 \times \eta$

$(L/T)^2 = g/k \tanh(kD)$; (similar to $c^2 = g/k \tanh(kD)$ for the progressive wave)

$q = H/2\ L/T \sin(\omega t) \sin(kx)$; [because: $\partial q/\partial x = - \partial y/\partial t = - \partial\eta/\partial t$]

$u = q\ k \cosh(kz) / \sinh(ky) = H/2\ L/T\ k \sin(\omega t) \sin(kx) \cosh(kz) / \sinh(ky)$; [because: $\int_0^y u\, dz = q$]

$w = \partial\eta/\partial t \sinh(kz) / \sinh(ky) = - H/2\ \omega \sin(\omega t) \cos(kx) \sinh(kz) / \sinh(ky)$

$p/\gamma = y - z + (G_s/g) (\cosh(ky) - \cosh(kz)) / (k \sinh(ky)) = D+\eta - z - \omega^2\ \eta/g/k \coth(ky)[1 - \cosh(kz)/\cosh(ky)]$

$= D - z + \eta \cosh(kz) / \cosh(ky)$; [because $\omega^2/g/k = \tanh(kD) = \tanh(ky)$ (1' order), and $\tanh(ky) \coth(ky) = 1$]

To prove the validity of the wave theory the formulas here will then for an ideal fluid be shown to fulfill the governing equations: **conservation of mass** and the **dynamic equations**, to the 1' order approximation:

$\partial u/\partial x + \partial w/\partial z = H/2\ k\ (L/T\ k - \omega) \sin(\omega t) \cos(kx) \cosh(kz) / \sinh(ky) = 0$ **conservation of mass** fulfilled

$G_z = dw/dt = \partial^2\eta/\partial t^2 \times \sinh(kz) / \sinh(ky) = - \omega^2 \times \eta \times \sinh(kz) / \sinh(ky)$; at the surface z = y: $G_s = \partial^2\eta/\partial t^2$

$- \partial p/\partial z = \gamma + \gamma\ (G_s/g) \sinh(kz) / \sinh(ky)$; unit weight $\gamma = \rho\ g$ unit mass × acceleration of gravity

$- \partial p/\partial z - \gamma = \rho\ G_z$ **vertical dynamic equation** fulfilled

$G_x = du/dt = H/2\ L/T\ k\ \omega \cos(\omega t)\sin(kx) \cosh(kz)/\sinh(ky) = - \partial\eta/\partial x\ L/T\ \omega \cosh(kz)/\sinh(ky) \tanh(ky)/\tanh(kD)$

$= - g\ \partial\eta/\partial x \cosh(kz)/(\cosh(ky)$ (using 1' order: $1= \tanh(ky)/\tanh(kD)$, $\tanh(kD) = L/T\ \omega/g$)

$\partial p/\partial x = \rho\ g\ \partial\eta/\partial x \cosh(kz)) / \cosh(ky)$; (with: $\gamma = \rho\ g$)

$- \partial p/\partial x = \rho\ G_x$ **horizontal dynamic equation** fulfilled

So this standing wave is in all the formulas a correct 1' order wave.

Below mean water level MWL, and below trough surface the wave pressure is:

$p^+/\gamma = \eta + (G_s/g) \times (\cosh(ky) - \cosh(kz)) / (k \sinh(ky))$

so as wanted: at trough surface $p^+/\gamma = \eta$, and at MWL below crest p^+/γ is less than η here in this wave description, a wave solution improved with better boundary conditions..

γ = weight of water 10 kN/m^3, g = acceleration of gravity = 9,81 m/sec^2 ≈ 10 m/sec^2

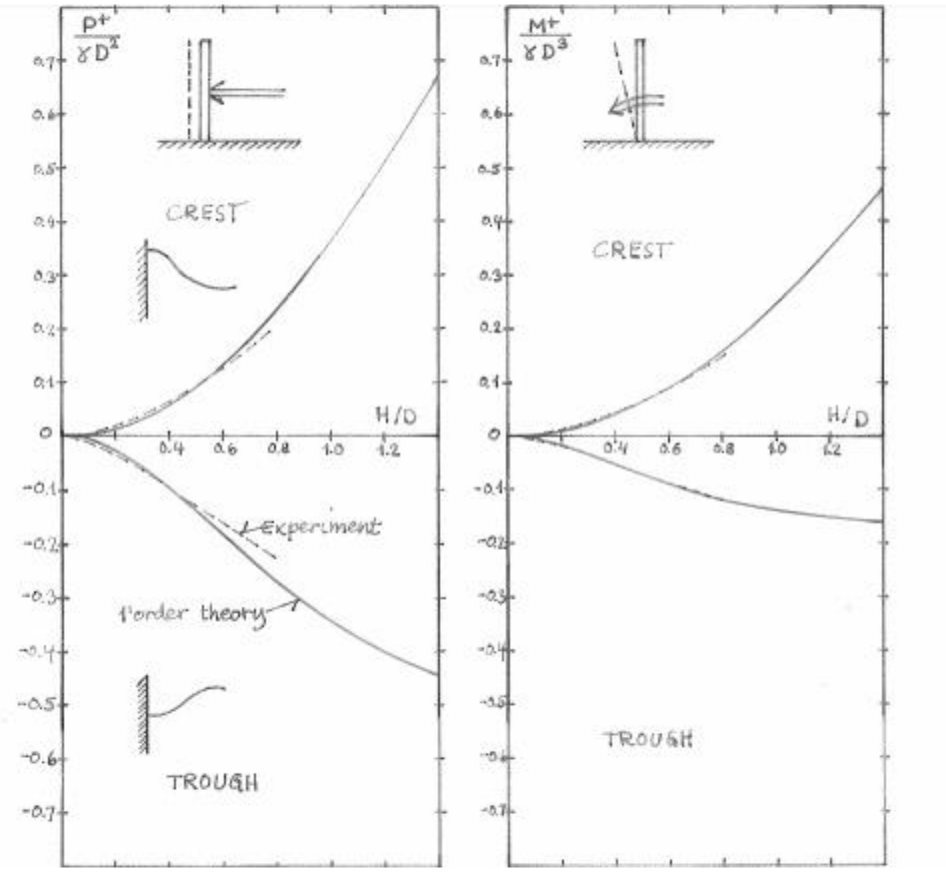


Figure 3: Wave pressure on a vertical wall from a standing wave of H/L = 16% steepness.
Using the 1′ order formulas written above and compared to my experiments 1968 at The Technical University of Denmark. In the model tests we measured the total horizontal sliding force and the overturning moment, needed for calculating the foundation stability of a breakwater.

We see that the theory is in reasonable good agreement with experiment.
The Airy wave formulas will not give quite as good agreement as can be seen of figure 2.
The difference is that in my formulas I use the actual water depth y (= D + η), and the Airy theory considers η so small negligible that the mean water depth D is used.
It is fully correct in a 1′ order wave theory to include 2′ order terms when we find it appropriate, and it should be done when it is in better agreement with reality.

In the 1800-s, using the mathematical potential theory (φ) with its assumption of non-rotational waves Airy developed the very good classical 1′order wave theory, and Stokes a 2′ order wave theory. Airy's mathematical wave theory for small waves has proved to be practical also for higher waves, and its simple formulas have been much used by engineers.

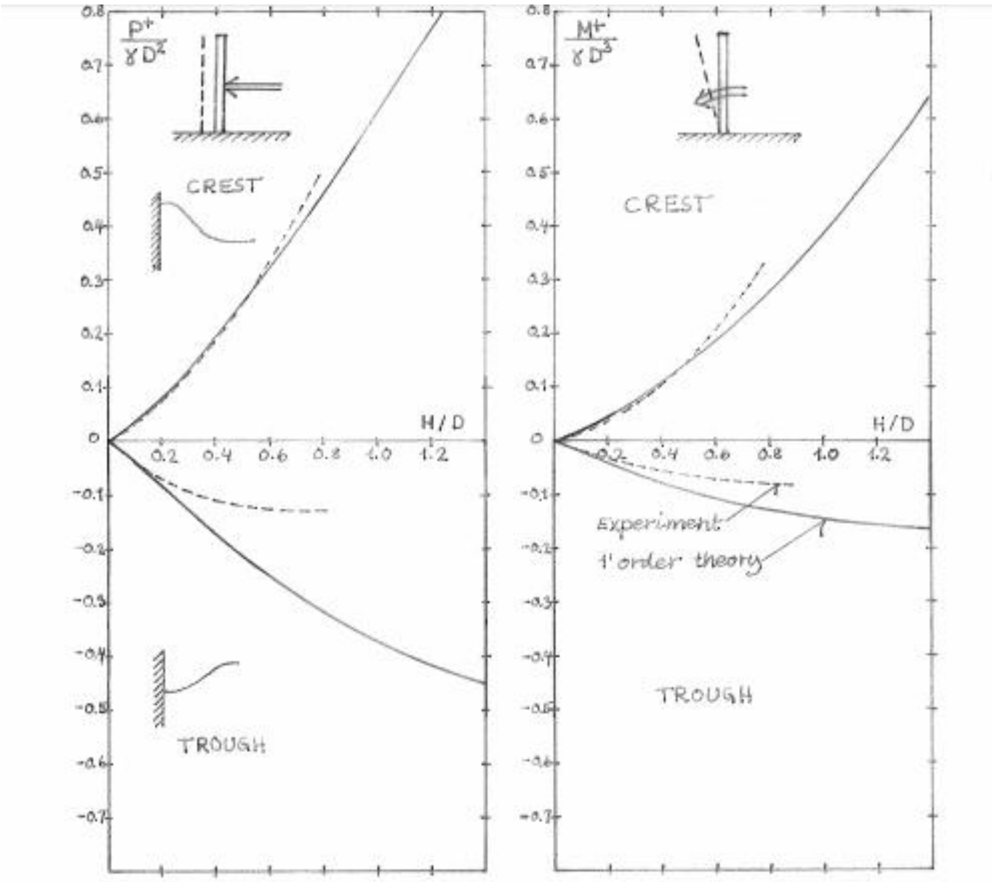
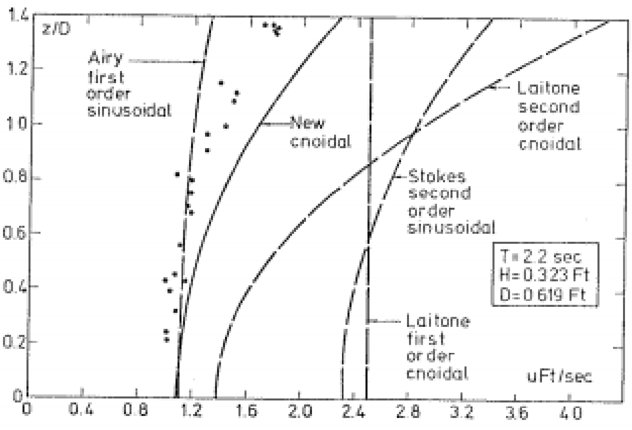


Figure 4: Wave pressure on a vertical wall from a standing wave of H/L = 3% steepness.

We see that our 1' order formula gives a too big negative pressure (the wave sucking force) below trough for this low steepness wave, shallow water wave.
(It is possible to make the shown deviating pressure result better by using practical 2' order alterations from e.g. the cnoi wave in some 1' order formulas – still in a fully correct 1' order wave theory).

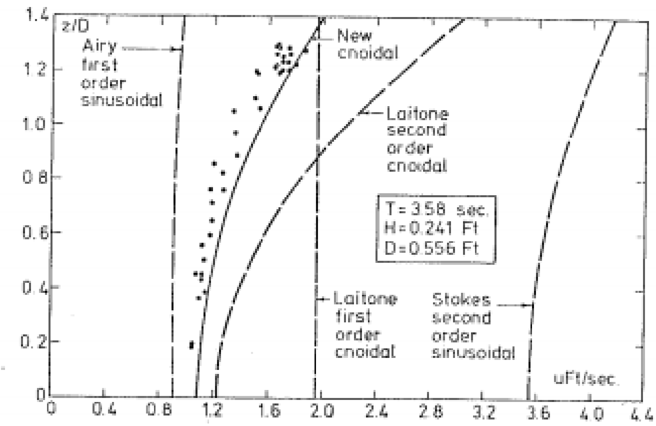
My similar formulas for the progressive wave on arbitrary depth can be proved in the same way.

(Airy' formula for horizontal velocity
$u = q \times R \times \cosh(Rz) / \sinh(RD) = c \times \eta \times 2\pi/L \times \cosh(Rz) / \sinh(RD)$
It is seen to give a forward water flow (wave flow) that can be substantial for progressive waves of practical height.)

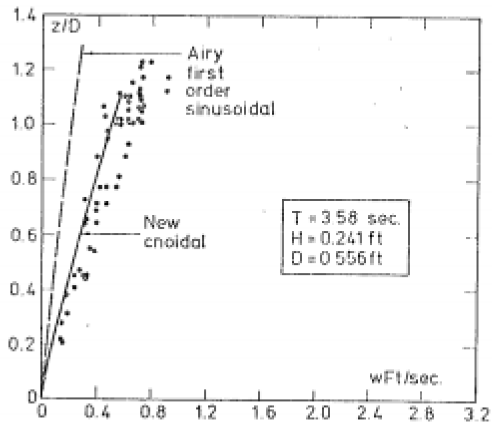


Sinusoidal calculations:

L_o = 7.6 metres
H/D = o.52
L/D = 16
H/L = 3.3%



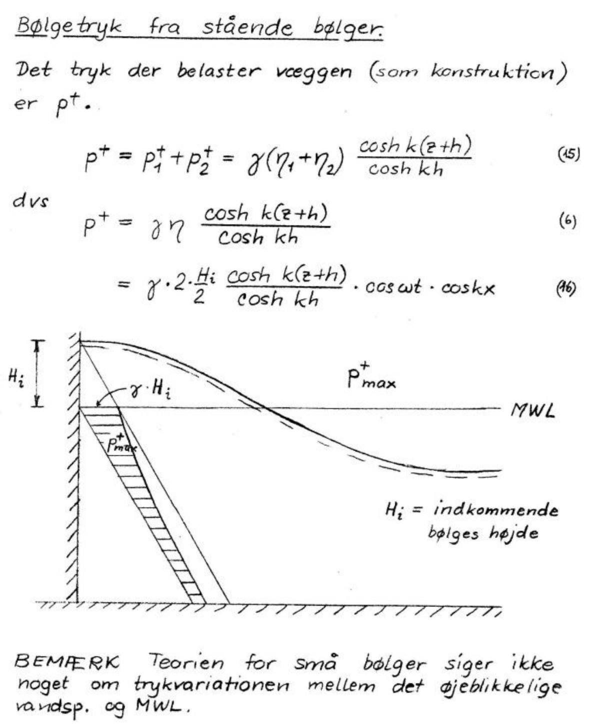
L_o = 2o metres
H/D = o.43
L/D = 27
H/L = 1.6%



Figs. 8, 9, and 1o.
Maximum horizontal and vertical particle velocities. Comparison of the cnoidal theory of this chapter with other theories and with experiments by Méhauté et al.

Figure 5: Measured water particle velocities in 2 progressive waves compared to wave theories.
This is a figure from my book showing Airy's simple 1' order formulas for maximum water particle velocities compared to some 2' order wave theories, and compared to laboratory experiments. We see how well Airy theory is for practical use. The measurements are seen to have been performed in a rather small wave flume. (I do not agree with the showing of Airy maximum vertical velocity above MWL z/D = 1).
In my book I have developed a 2' order cnoidal wave theory to be used from infinite depth all the way to shallow water solitary wave.

APPENDIX: Wave pressure on a vertical wall according to the traditional potential wave theory?



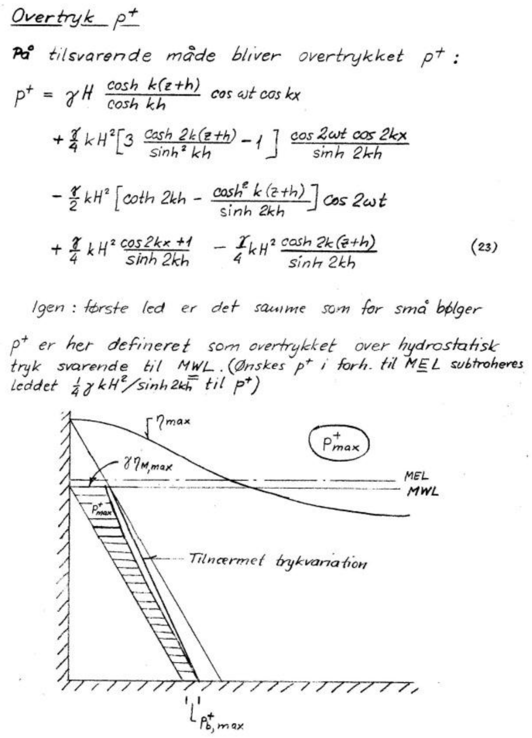


Figure 6: Wave pressure – disregarding Newton's 2' law (of momentum)?

The figures here are from Danish Technical University, ISVA, DTU, 1973-1974, from page 7 and 14, used for education of wave pressure according to the potential theory of 1' order (to the left) and 2' order (to the right), showing hydrostatic pressure in the wave top – in contrast to the better formula given on page 3?

Improved basis for practical use

For the engineer designing harbor constructions there is an obvious statistical variation of all wave phenomena. Statistical variation is also the case for e.g. material parameters for the concrete used in the harbor breakwater. For the practical use of concrete an agreed "allowed" design strength is given to the engineer, based on scientific experiments and statistical evaluation, with the statistical rare risk that for your construction the concrete happens to be of lower value in strength than declared. It may be felt that the wave force is so unpredictable that it is unimportant what formula to use, because the statistical variation may give a bigger deviation than the correction using the formulas presented here. But no. Instead of using the classic potential theory the engineer's decision for choosing a practical design force using 1' order waves should be based on the obvious better theoretical solution with the simple mathematical better formulas here.

Niels Mejlhede Jensen, 2022.11.29