CHAPTER VII

PROGRESSIVE AND STANDING SECOND ORDER SINUSOIDAL WAVES

ABSTRACTS

In this chapter the basic method applied to give the first order waves of chapter IV and V will be expanded to give also different types of the sinusoidal waves of second order.

The second order waves on arbitrary depth will be more complicated than the second order deep water waves found in chapter VI. The wave is not assumed to be irrotational, but if at the end the rotation is set equal to zero the wave will be the same as the Stokes' second order wave, as far as the surface profile is concerned.

INTRODUCTION

During the development of the theory for the first order waves of chapters IV and V we had to neglect some terms. In chapter VI we saw for deep water waves that we could improve the theory rather much by including the second order terms. So we will also do that here. The progressive wave can be found more easy alone, but we saw in chapter V that with little extra work we can as well find both the progressive and the standing wave at the same time.

BASIC EQUATIONS

We consider two dimensional gravity waves on water with horizontal bottom.

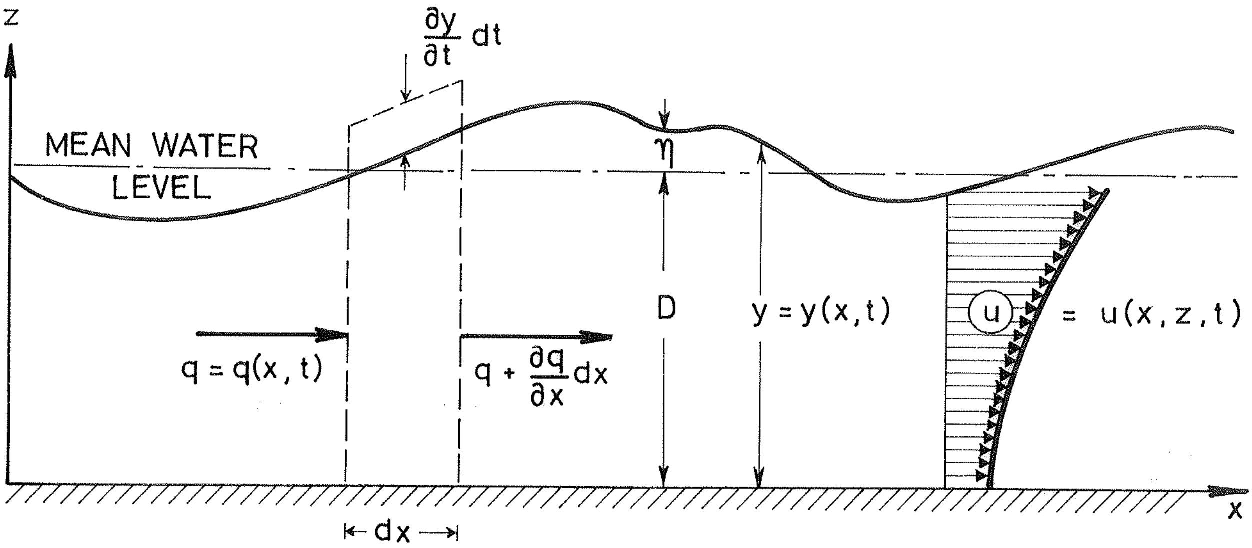

Fig. 1. Definition sketch

From the definition sketch we see that

$$y = D + \eta \tag{1}$$

and we find the equation of continuity

$$\frac{\partial q}{\partial x} = -\frac{\partial y}{\partial t} = -\frac{\partial \eta}{\partial t} \tag{2}$$

$q = q(x,t)$ is the discharge through a vertical, x is the horizontal co-ordinate, t is the time, c is the wave celerity, $y = y(x,t)$ the actual water depth, and $\eta = \eta(x,t)$ is the surface elevation, D is the mean water depth.

q can also be found by the integration of u

$$q = \int_0^y u \, dz \tag{3}$$

The distribution of u over a vertical is an unknown function, $f(x,z,t)$. z is the vertical coordinate with $z = o$ at the bottom. We will here express $f(x,z,t)$ as a series of hyperbolic function, so we write

$$u = \sum q_i R_i \frac{\cosh R_i z}{\sinh R_i y} \tag{4}$$

where R in the beginning is considered a constant with the dimension of a reciprocal length. We will then investigate one of the terms of eq. 4.

$$u = qR \frac{\cosh Rz}{\sinh Ry} \tag{5}$$

From chapter IV we know that this term alone will give us the first order theory with $R=2\pi/L$, L being the wave length. But we must not forget eq. 4. For the deep water wave in chapter VI it was possible to use the expression corresponding to eq. 5 also to find second order waves. For the waves on arbitrary depth of this chapter here it will not be quite so easy. Even for the second order sinusoidal wave it will be necessary to use eq. 4 with two different terms, and for the third order sinusoidal theory, eq. 4 will be involved with several different terms. This is though not so difficult as it sounds, because it all comes out automatically, it is felt, of a theory based on only eq. 5.

For the second order cnoidal wave of chapter IX eq. 5 alone will be satisfactory, which is a big advantage, but then R turns out to be a function of η. The hyperbolic form in eq. 4 for $f(x,z,t)$ has of course been chosen because it is convenient. The hyperbolic form is known from experiments, from the classical Airy theory, and from chapters IV and V. At the infinite depth limit it will give the exponential form, known from chapter VI.

Using the equation of continuity

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} \tag{6}$$

and $w = o$ at the bottom, $z = o$, and eq. 2 the vertical particle velocity $w = w(x,z,t)$ is found from eq. 5 to

$$w = \frac{\partial \eta}{\partial t}\frac{\sinh Rz}{\sinh Ry} + q\frac{\partial \eta}{\partial x} R \frac{\coth Ry \sinh Rz}{\sinh Ry} \tag{7}$$

The horizontal particle acceleration $G_x = G_x(x,z,t)$ is now found from eqs. 4 and 7

$$G_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \frac{\partial q}{\partial t} R \frac{\cosh Rz}{\sinh Ry}$$

$$+ q\frac{\partial q}{\partial x} R^2 \frac{\cosh Ry \cosh Rz + 1}{\sinh^2 Ry} - q^2 \frac{\partial \eta}{\partial x} R^3 \frac{\coth Ry}{\sinh^2 Ry} \tag{8}$$

In eq. 8 the three terms are of different magnitude. $\frac{\partial q}{\partial t}$ is small of first order, $q\frac{\partial q}{\partial x}R$ of second order and $q^2\frac{\partial \eta}{\partial x}R^2$ of third order. Comparison of the magnitudes of the different terms can be made with the solutions obtained later. The third order terms will here be considered negligible, so that e.g. the last term in eq. 8 will be neglected.

The vertical particle acceleration $G_z = G_z(x,z,t)$ is found in the same way as eq. 8 by

$$G_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \tag{9}$$

The vertical dynamic equation for a frictionless fluid,

$$-\frac{\partial p}{\partial z} - \varrho g = \varrho G_z \tag{10}$$

where ϱ and g are the unitmass and the acceleration of gravity, is used together with $p = 0$ at the surface $z = y$, to obtain an expression for the pressure $p = p(x,z,t)$

$$\frac{p}{\gamma} = y - z + \frac{1}{g}\Bigg\{\frac{\partial^2\eta}{\partial t^2}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry}$$

$$+\left[-\left(\frac{\partial \eta}{\partial t}\right)^2 + \frac{\partial q}{\partial t}\frac{\partial \eta}{\partial x} + q\frac{\partial^2 \eta}{\partial x \partial t}\right]\frac{\coth Ry\,[\cosh Ry - \cosh Rz]}{\sinh Ry}$$

$$+\left[\left(\frac{\partial q}{\partial x}\right)^2 - q\frac{\partial^2 q}{\partial x^2}\right]\frac{1}{4}\frac{\cosh 2Ry - \cosh 2Rz}{\sinh^2 Ry}\Bigg\} \tag{11}$$

where again third and higher order terms are neglected. By differentation of eq. 11 an expression for $\frac{\partial p}{\partial x}$ is found. Through the horizontal dynamic equation

$$-\frac{\partial p}{\partial x} = \varrho G_x \tag{12}$$

an alternative expression for $\frac{\partial p}{\partial x}$ is found.

Eliminating $\frac{\partial p}{\partial x}$ from the two equations a wave equation of second order is derived

$$g\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x \partial t^2}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} + \frac{\partial q}{\partial t}R\frac{\cosh Rz}{\sinh Ry}$$

$$+\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial t^2} + \left[-2\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial t^2} - 3\frac{\partial \eta}{\partial t}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\partial q}{\partial t}\frac{\partial^2 \eta}{\partial x^2} + q\frac{\partial^3 \eta}{\partial x^2 \partial t}\right]$$

$$\frac{\coth Ry\,[\cosh Ry - \cosh Rz]}{\sinh Ry} - q\frac{\partial \eta}{\partial t}R^2\frac{\cosh Ry \cosh Rz + 1}{\sinh^2 Ry}$$

$$+\left[\frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial x^2} - q\frac{\partial^3 q}{\partial x^3}\right]\frac{1}{4}\frac{\cosh 2Ry - \cosh 2Rz}{\sinh^2 Ry} = 0 \qquad (13)$$

FIRST ORDER SOLUTION

Eq. 13 can be used to find sinusoidal solutions of first and second order for progressive and standing waves. To find the first order solutions we neglect the second order terms in eq. 13, so the wave equation will reduce to

$$g\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x \partial t^2}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} + \frac{\partial q}{\partial t}R\frac{\cosh Rz}{\sinh Ry} = 0 \qquad (14)$$

We will here shortly give the first order solutions already found in chapters IV and V.

By the use of eq. 1 coth Ry can be expanded as

$$\coth Ry = \coth R(D+\eta) \simeq \coth RD - \frac{R\eta}{\sinh^2 RD} \qquad (15)$$

so that in eq. 14 we can make the substitution $\coth Ry \simeq \coth RD$. Eq. 14 is then differentiated with respect to x and eq. 2 is used to eliminate q. This equation is then split into two equations, one equation of the z-dependent terms

$$\frac{\partial^4 \eta}{\partial x^2 \partial t^2}\frac{1}{R}\frac{\cosh Rz}{\sinh Ry} + \frac{\partial^2 \eta}{\partial t^2}R\frac{\cosh Rz}{\sinh Ry} = 0 \qquad (16)$$

and one equation of the z-independent terms

$$g\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^4 \eta}{\partial x^2 \partial t^2}\frac{1}{R}\coth RD = 0 \qquad (17)$$

and we then want these two equations to be fulfilled simultaneously. One such solution is the wellknown first order progressive wave

$$\eta = \eta_1 = \frac{H}{2}\cos k(x-ct) \qquad \text{with} \qquad R = k = \frac{2\pi}{L} \tag{18}$$

$$c = \frac{L}{T} = \sqrt{\frac{g}{k}\tanh kD} \tag{19}$$

with a solution to q that fulfils eq. 2

$$q = q_1 = c\eta = c\frac{H}{2}\cos k(x-ct) \tag{2o}$$

Eqs. 18, 19, and 2o are seen to fulfil eq. 14.

Another solution to eqs. 16 and 17 is the first order standing wave

$$\eta = \eta_1 = \frac{H}{2}\cos\omega t\cos kx \qquad \text{with} \qquad R = k = \frac{2\pi}{L} \tag{21}$$

$$\frac{\omega}{k} = \frac{L}{T} = \sqrt{\frac{g}{k}\tanh kD} \tag{22}$$

and fulfilling eq. 2

$$q = q_1 = \frac{H}{2}\frac{\omega}{k}\sin\omega t\,\sin kx \tag{23}$$

We can now make an approximate evaluation of the magnitude of the terms in eq. 13 by insertion of the first order solutions for η and q, to see if the second order terms are negligible. We compare $3\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial x\partial t}$ with $\frac{\partial q}{\partial t}R$ using eqs. 18, 19, and 2o and get

$$3\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial x\partial t}\bigg/\frac{\partial q}{\partial t}R = 3\pi\frac{H}{L}\cos k(x-ct) \tag{24}$$

From this is seen that H/L must then be unrealistically small for this second order term to be negligible. Comparison of two other terms in eq. 13 or in eq. 15 would demand also H/D to be small. Hence it would be of interest to develop a more satisfactory theory, as attempted later.

SECOND ORDER PROGRESSIVE SOLUTION

Now the whole second order wave equation, eq. 13 is used, with the hope of making a better theory.

For a progressive wave of permanent form we have

$$-\frac{\partial \eta}{\partial t} = c\frac{\partial \eta}{\partial x} \tag{25}$$

and

$$q = c\eta \tag{26}$$

Like for the deep water wave in chapter VI we can expect a solution of the type

$$\eta = \eta_1 + \eta_2 \tag{27}$$

$$q = q_1 + q_2 \tag{28}$$

where η_1 and q_1 are the already given first order expressions eqs. 18 and 2o. η_2 and q_2 are second order correction terms to be found. It is seen that when substituting η and q from eqs. 27 and 28 into the wave equation, eq. 13, the unknown η_2 and q_2 will give only third and fourth order contributions in the second order terms of eq 13, so that the second order terms of eq. 13 can be substituted only with η_1 and q_1, the first order solutions of eqs. 18 and 2o. The equation is then split into a z-dependent equation

$$c^2\frac{\partial^3 \eta}{\partial x^3}\frac{1}{k}\frac{\cosh kz}{\sinh ky} + c^2\frac{\partial \eta}{\partial x}k\frac{\cosh kz}{\sinh ky} = -3c^2\left(\frac{H}{2}\right)^2 k^3 \coth kD\,\frac{\cosh kz}{\sinh ky}\sin 2k(x-ct) \tag{29}$$

and a z-independent equation

$$g\frac{\partial \eta}{\partial x} + c^2\frac{\partial^3 \eta}{\partial x^3}\frac{1}{k}\coth ky = c^2\left(\frac{H}{2}\right)^2 k^3\left[4\coth^2 kD - 1\right]\sin 2k(x-ct) \tag{3o}$$

Let us first consider eq. 29. Substituting eq. 27 with η_1 from eq. 18 into eq. 29 we get

$$\frac{\partial^3\eta_2}{\partial x^3}\frac{1}{R}\frac{\cosh Rz}{\sinh Ry}+\frac{\partial\eta_2}{\partial x}R\frac{\cosh Rz}{\sinh Ry}=$$

$$-3\left(\frac{H}{2}\right)^2k^3\coth kD\frac{\cosh kz}{\sinh ky}\sin 2k(x-ct) \qquad (31)$$

because η_1 with R = k was a solution to the left side alone. (This was simply how η_1 originally was found).

Another solution to the left side alone of eq. 31 would be

$$\eta_2 \simeq \cos R(x-ct) \qquad (32)$$

for any R. Eq. 32 will be used in a moment. A solution that takes care of the right side of eq. 31 is now of interest. On the left side we have the variable cosh Rz and on the right side cosh kz. This can only be fulfilled for any z for R = k. We then get the solution

$$\eta_2=\eta_{2a}=\left(\frac{H}{2}\right)^2\frac{k}{2}\coth kD\cos 2k(x-ct) \quad \text{with} \quad R=k \qquad (33)$$

We then turn our attension to eq. 3o. Besides η_{2a} we may have any solution of the type shown in eq. 32, let us call it η_{2b}, so

$$\eta=\eta_1+\eta_{2a}+\eta_{2b} \qquad (34)$$

coth Ry is substituted by eq. 15, and substituting η in eq.3o by eq. 34, using eqs. 18 and 33, we get

$$g\frac{\partial\eta_{2b}}{\partial x}+c^2\frac{\partial^3\eta_{2b}}{\partial x^3}\frac{1}{R}\coth RD=$$

$$\left(\frac{H}{2}\right)^2\frac{3}{2}k^3\frac{1}{\sinh^2 kD}\sin 2k(x-ct) \qquad (35)$$

Because of 2k in the argument of sin on the right side, R in eq. 32 must be R = 2k, so we get for η_{2b}

$$\eta_2=\eta_{2b}=\left(\frac{H}{2}\right)^2\frac{3}{4}k\frac{\coth kD}{\sinh^2 kD}\cos 2k(x-ct) \quad \text{with} \quad R=2k \qquad (36)$$

In eq. 35 we used c from eq. 19, which gives

$$\frac{g}{c^2} = k \coth kD \tag{37}$$

We could instead have proposed

$$\frac{g}{c^2} = k \coth kD + \alpha k H \tag{38}$$

so that the celerity would depend on the wave height. But such a proposal would demand a second order term on the right side of eq. 3o containing sin k(x - ct). So we end up with only eq. 37.

We have then found the solution for the progressive second order wave, eq. 34 with eqs. 18, 33 and 36.

When describing only the surface profile η_{2a} and η_{2b} can as well be combined into one expression, η_2. But this may not be practical, because R is different for η_{2a} and η_{2b}. This must be remembered when going back to get the second order values of u, w, and p from eqs 5, 7, and 11. With eqs. 5, 26, 34, 18, 33, and 36 u will be

$$u = c\left\{\eta_1 k \frac{\cosh kz}{\sinh ky} + \eta_{2a} k \frac{\cosh kz}{\sinh ky} + \eta_{2b} 2k \frac{\cosh 2kz}{\sinh 2ky}\right\} \tag{39}$$

w and p are written likewise, see the appendix. Eq. 39 is illustrated in fig. 2.

The solution eq. 34 is reasonable for deep water. But for more shallow water η_{2b} in eq. 36 does not seem to be a reasonable second order correction to the first order wave because 'the small η_{2b} becomes so big'. Instead of having second order waves on top of a first order wave it is possible to get a solution where the first order wave is deformed by a second order correction term in the argument of the cosine function

$$\eta = \frac{H}{2} \cos\left(k(x-ct) - \alpha \frac{\partial \eta}{\partial x}\right) + \Delta D \tag{4o}$$

ΔD is determined from the definition of the mean water level, $\int_0^L \eta \, dx = 0$. α is found to

$$\alpha = k \coth kD \left[1 + \frac{3}{2} \frac{1}{\sinh^2 kD}\right] \tag{41}$$

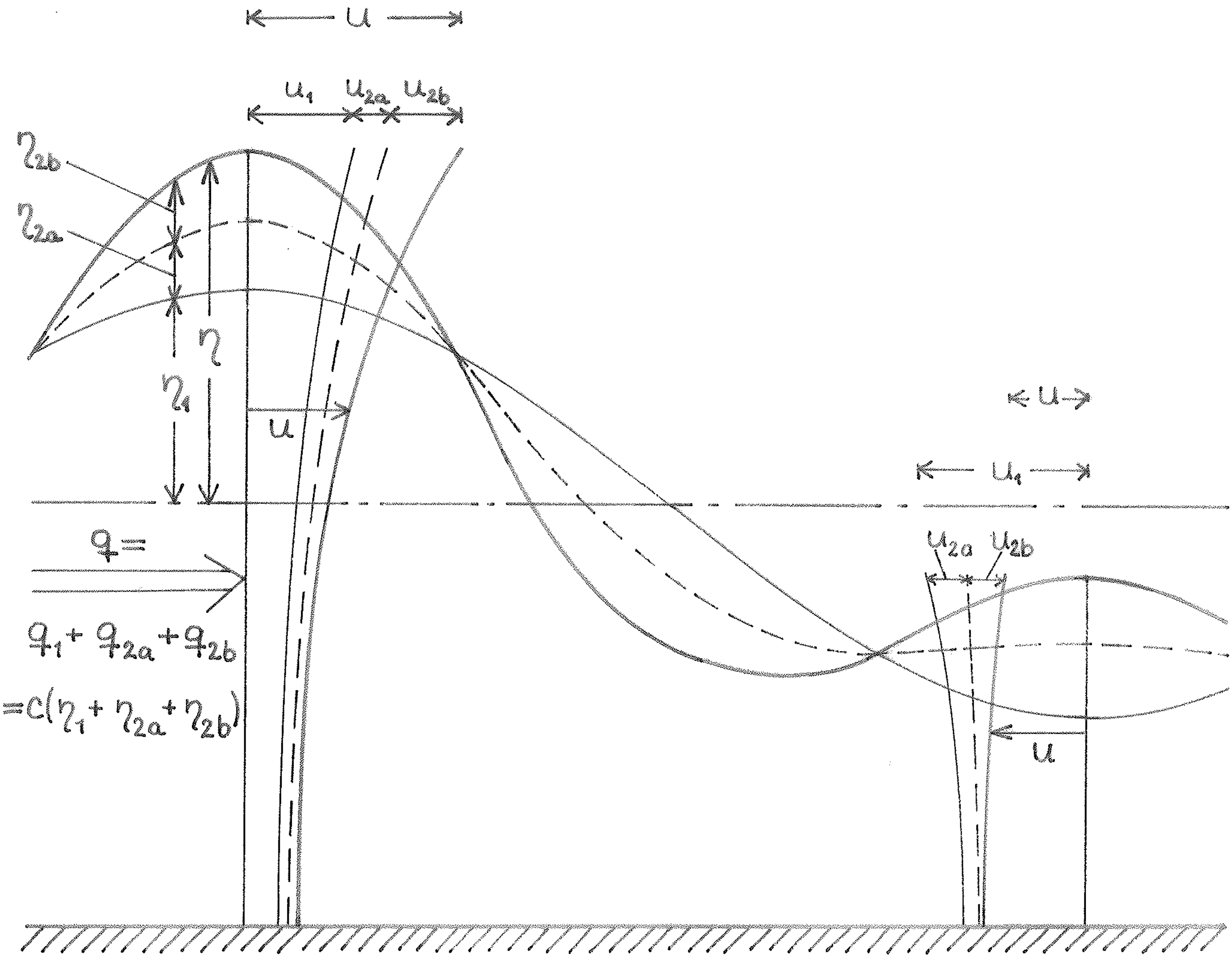

Fig. 2 Horizontal velocity profile in a second order sinusoidal wave. The surface profile consists of $= \eta_1 + \eta_{2a} + \eta_{2b}$ giving $q = q_1 + q_{2a} + q_{2b}$. u_1 and u_{2a} are vertically distributed as cosh kz, while u_{2b} is distributed as cosh 2kz. If a cosh Rz expression is written for the final u, then R must be a variable, so that R for the crest is between k and 2k, and R for the trough is less than k. It seems as if $R = R(\eta)$, (leading to the cnoidal wave of chapter IX).

Within the frames of a second order theory the first order expression for $\frac{\partial \eta}{\partial x}$ can be substituted into eq. 4o to give

$$\eta = \frac{H}{2}\cos\theta_s + \triangle D \tag{42}$$

with

$$\theta_s = k(x-ct) + \alpha \frac{H}{2} k \sin\theta_s \tag{43}$$

The solutions eqs. 34, 4o, and 42, show that a second order wave can be many things. They all fulfil the wave equation, eq. 12, to the same degree of approximation. But they give different surface profiles, because they have different ways of including 'hidden' higher order terms. But none of the solutions are good for more shallow water. So it will be appropriate to propose still another second order solution : the cnoidal wave on arbitrary depth. In mathematical respects it will be of the same degree of approximation as other second order solutions. But in practical respects it will be significantly better. (See chapter IX)

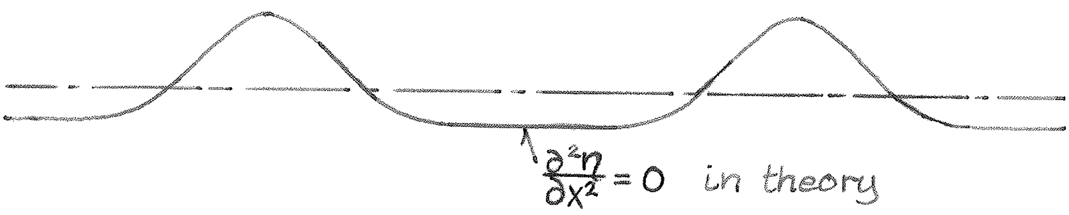

Fig. 3 The second order 'argument wave' can only 'flatten the trough' to a certain point, when $\partial^2\eta/\partial x^2$ in the trough gets to be = o.

SECOND ORDER STANDING SOLUTION

The solution to the standing wave is found in a similar way as the above progressive solution, so we will make this more short. We expect solutions of the type given in eqs. 27 and 28. The first order solutions η_1 and q_1 from eqs. 21 and 23 are substituted into the second order wave equation, eq. 13.

The z-dependent equation for η_2, corresponding to eq. 31 will then be

$$\frac{\partial^3 \eta_2}{\partial x \partial t^2} \frac{1}{R} \frac{\cosh Rz}{\sinh Ry} - \frac{\partial q_2}{\partial t} R \frac{\cosh Rz}{\sinh Ry}$$
$$= \frac{3}{2}\left(\frac{H}{2}\right)^2 \omega^2 k \coth kD \frac{\cosh kz}{\sinh ky} \cos \omega t \sin kx \qquad (44)$$

The term $\frac{\partial^3 \eta_2}{\partial x \partial t^2}$ can through eq. 2 be substituted by $-\frac{\partial^3 q_2}{\partial x^2 \partial t}$.
We then find the solution

$$q_2 = q_{2a} = \left(\frac{H}{2}\right)^2 \frac{\omega}{4} \coth kD \sin 2\omega t \sin 2kx \text{ with } R=k \qquad (45)$$

Corresponding to q_{2a} we have through eq. 2

$$\eta_{2a} = \left(\frac{H}{2}\right)^2 \frac{k}{4} \coth kD \cos 2\omega t \cos 2kx \text{ with } R=k \qquad (46)$$

Like for the progressive wave we then consider the z-independent equation. Corresponding to eq. 36 we then find the solution

$$\eta_{2b} = \left(\frac{H}{2}\right)^2 \frac{k}{4}\left[\tanh kD + \frac{1+3\coth^2 kD \cos 2\omega t}{\sinh 2kD}\right] \cdot$$
$$\cos 2kx \qquad \text{with } R = 2k \qquad (47)$$

and through eq. 2

$$q_{2b} = \left(\frac{H}{2}\right)^2 \frac{3}{4}\omega \frac{\coth^2 kD}{\sinh 2kD} \sin 2\omega t \sin 2kx$$
$$\text{with } R = 2k \qquad (48)$$

The final solution for η for the standing wave is then

$$\eta = \eta_1 + \eta_{2a} + \eta_{2b} \tag{49}$$

and for q

$$q = q_1 + q_{2a} + q_{2b} \tag{5o}$$

Eqs. 49 and 5o could now be tested in the second order wave equation, eq. 13.

The horizontal particle velocity u for the standing wave is then found from eq. 5 to

$$u = q_1 k \frac{\cosh kz}{\sinh ky} + q_{2a} k \frac{\cosh kz}{\sinh ky} + q_{2b} 2k \frac{\cosh 2kz}{\sinh 2ky} \tag{51}$$

where q_1, q_{2a}, and q_{2b} are substituted from eqs. 23, 45, and 48. The important thing to remember is that R = 2k in the last term.

The expressions for w and p are found in the same way from eqs. 7 and 11.

It is also possible for the standing wave to find a solution with the second order correction term in the argument, like eqs. 4o and 42.

$$\eta = \frac{H}{2} \cos \Theta_t \cos \Theta_x + \eta_p + \Delta D \tag{52}$$

where ΔD is determined from the definition of the mean water level, $\int_0^L \eta \, dx = o$, and where Θ_t, Θ_x, and η_p are

$$\Theta_t = \omega t + \alpha \frac{H}{4} \sin \omega t \cos kx \tag{53}$$

$$\Theta_x = kx + \alpha \frac{H}{4} \cos \omega t \sin kx \tag{54}$$

$$\eta_p = \left(\frac{H}{2}\right)^2 \frac{k}{4} \coth 2kD \cos 2kx \tag{55}$$

α is the same as found for the progressive wave in eq. 41

$$\alpha = k \coth kD \left[1 + \frac{3}{2} \frac{1}{\sinh^2 kD}\right] \tag{56}$$

q is found to

$$q = \frac{H}{2}\frac{L}{T}\sin\theta_t \sin\theta_x \tag{57}$$

The above standing wave is described more in detail by the author in 1975.

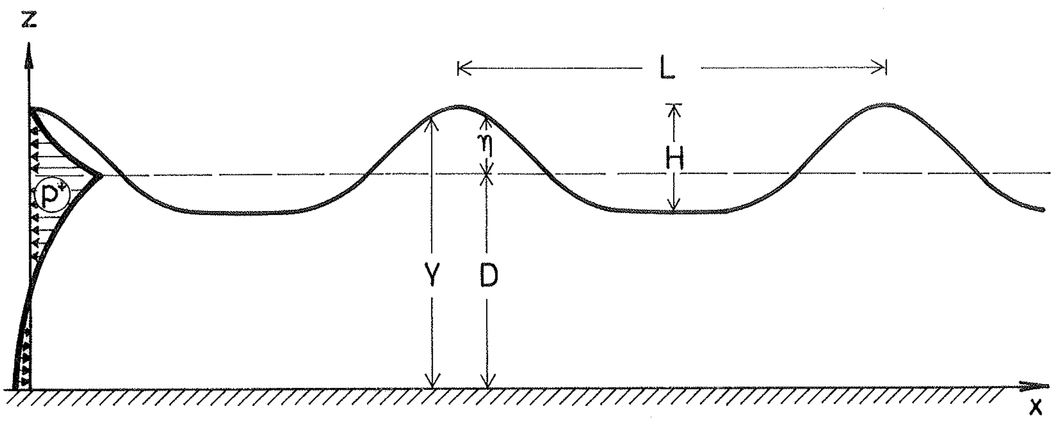

Fig. 4 The standing second order sinusoidal wave with the correction in the argument, eq. 52. The influence of the vertical acceleration on the pressure on the vertical wall can be so big in a second order wave that the pressure below the crest, near the bottom will be negative, which is also known from experiments. But the sinusoidal second order theory may for higher waves give unreasonable big negative pressures, as well for the traditional theories as for the theory of this chapter.

APPENDIX I

FINAL FORMULAS FOR PROGRESSIVE AND STANDING SECOND ORDER SINUSOIDAL WAVES

L/T, u, w, and p can be written with the same expressions for both the progressive and the standing wave.

$$k = \frac{2\pi}{L} \tag{58}$$

$$\omega = \frac{2\pi}{T} \tag{59}$$

$$c = \frac{L}{T} = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kD} \tag{60}$$

$$u = u_1 + u_{2a} + u_{2b} = (q_1 + q_{2a})\, k \frac{\cosh kz}{\sinh ky} + q_{2b} \cdot 2k \frac{\cosh 2kz}{\sinh 2ky} \tag{61}$$

$$w = w_1 + w_{2a} + w_{2b} = \left[\frac{\partial \eta_1}{\partial t} + \frac{\partial \eta_{2a}}{\partial t}\right] \frac{\sinh kz}{\sinh ky} + \frac{\partial \eta_{2b}}{\partial t} \frac{\sinh 2kz}{\sinh 2ky} + q_1 \frac{\partial \eta_1}{\partial x} k \coth ky \frac{\sinh kz}{\sinh ky} \tag{62}$$

$$\frac{p}{\gamma} = \frac{p_1}{\gamma} + \frac{p_{2a}}{\gamma} + \frac{p_{2b}}{\gamma} = D - Z + \eta$$
$$+ \frac{1}{g}\Bigg\{\left[\frac{\partial^2 \eta_1}{\partial t^2} + \frac{\partial^2 \eta_{2a}}{\partial t^2}\right] \frac{1}{k} \frac{\cosh ky - \cosh kz}{\sinh ky}$$
$$+ \frac{\partial^2 \eta_{2b}}{\partial t^2} \frac{1}{2k} \frac{\cosh 2ky - \cos 2kz}{\sinh 2ky}$$
$$+ \left[-\left(\frac{\partial \eta_1}{\partial t}\right)^2 + \frac{\partial q_1}{\partial t} \frac{\partial \eta_1}{\partial x} + q_1 \frac{\partial^2 \eta_1}{\partial x \partial t}\right] \coth ky \cdot$$
$$\frac{\cosh ky - \cosh kz}{\sinh ky}$$
$$+ \left[\left(\frac{\partial \eta_1}{\partial t}\right)^2 - q_1 \frac{\partial^2 q_1}{\partial x^2}\right] \frac{1}{4} \frac{\cosh 2ky - \cosh 2kz}{\sinh^2 ky}\Bigg\} \tag{63}$$

PROGRESSIVE WAVE

$$\eta = \eta_1 + \eta_{2a} + \eta_{2b} = \frac{H}{2} \cos k(x - ct)$$
$$+ \left(\frac{H}{2}\right)^2 \frac{k}{2} \coth kD \ \cos 2k(x - ct)$$
$$+ \left(\frac{H}{2}\right)^2 \frac{3}{4} k \frac{\coth kD}{\sinh^2 kD} \cos 2k(x - ct) \tag{64}$$

$$q = q_1 + q_{2a} + q_{2b} = c\,(\eta_1 + \eta_{2a} + \eta_{2b}) \tag{65}$$

STANDING WAVE

$$\eta = \eta_1 + \eta_{2a} + \eta_{2b} = \frac{H}{2} \cos \omega t \ \cos kx$$
$$+ \left(\frac{H}{2}\right)^2 \frac{k}{4} \coth kD \ \cos 2\omega t \ \cos 2kx$$
$$+ \left(\frac{H}{2}\right)^2 \frac{k}{4} \left[\tanh kD + \frac{1 + 3\coth^2 kD \ \cos 2\omega t}{\sinh 2kD}\right] \cos 2kx \tag{66}$$

$$q = q_1 + q_{2a} + q_{2b} = \frac{H}{2} \frac{L}{T} \sin \omega t \ \sin kx$$
$$+ \left(\frac{H}{2}\right)^2 \frac{\omega}{4} \coth kD \ \sin 2\omega t \ \sin 2kx$$
$$+ \left(\frac{H}{2}\right)^2 \frac{3}{4} \omega \frac{\coth kD}{\sinh 2kD} \sin 2\omega t \ \sin 2kx \tag{67}$$

The 'argument wave' is given in eqs. 4o, 41, and 42 for the progressive wave, and in eqs. 52, 53, 54, 55, 56, and 57 for the standing wave.

NUMERICAL EXAMPLE

Let us consider the progressive wave with the period, wave height, and mean water depth given as, (the same wave as used in chapter IV)

$$T = 1o \text{ seconds}; \quad H = 6 \text{ metres}; \quad D = 1o \text{ metres}. \tag{68}$$

We find

$$L_o = \frac{g}{2\pi} T^2 = 1.56 \cdot 1o^2 = 156\text{m} \tag{69}$$

The tables for Airy waves give

$$L = 93 \text{ m} \tag{7o}$$

The wave profile can then be calculated to give the following maximum values

$$\left(\frac{\eta_{2a}}{\eta_1}\right)_{crest} = \frac{1}{2}\pi \frac{H}{L} \coth kD = o.18 \tag{71}$$

$$\left(\frac{\eta_{2b}}{\eta_1}\right)_{crest} = \frac{3}{4}\pi \frac{H}{L} \frac{\coth kD}{\sinh^2 kD} = o.49 \tag{72}$$

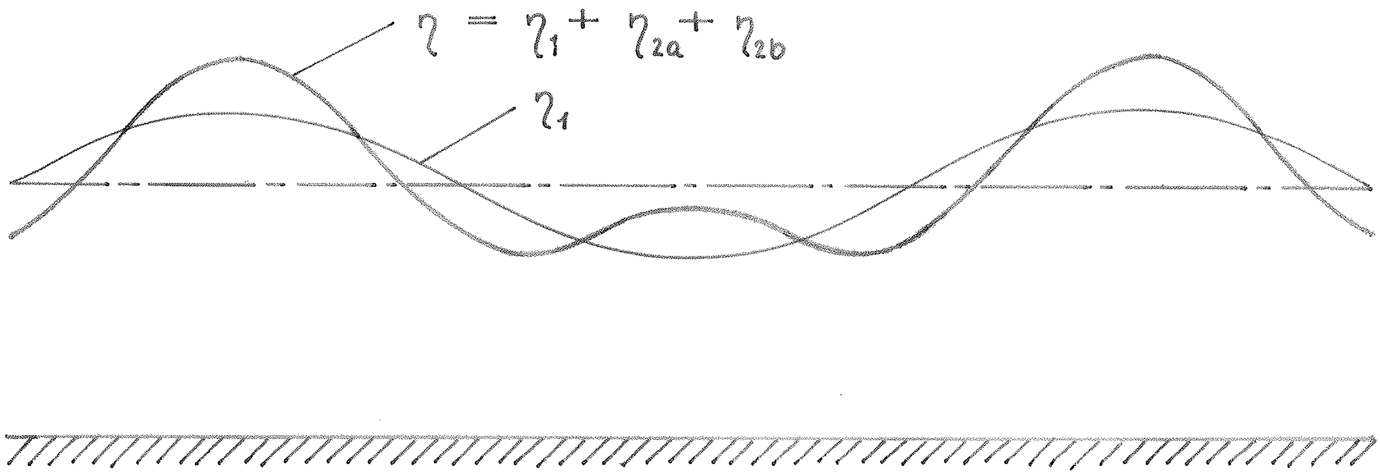

Fig. 5. The wave profile for the realistic example of this appendix, according to the second order theory of this chapter (or according to Stokes' theory). We see that we can hardly use this profile and should try to find a better theory.

STOKES' THEORY

We will here show that the second order sinusoidal theory of this chapter for irrotational motion is identical within usual second order approximations with the classical Stokes' theory.

The surface profile is given by eq. 64 and we see right away that this expression is identical with the Stokes' expression.

For u, w, and p it is necessary with somewhat longer considerations.

The Stokes' expressions for u, w, and p^+ are

$$u = c\,\frac{H}{2}\,k\,\frac{\cosh kz}{\sinh kD}\cos\theta + \frac{3}{16}\,c\,H^2k^2\,\frac{\cosh 2kz}{\sinh^4 kD}\cos 2\theta$$

$$- \frac{1}{8}\,c\,H^2k^2\,\frac{\coth kD}{kD} \tag{73}$$

$$w = -\,c\,\frac{H}{2}\,k\,\frac{\sinh kz}{\sinh kh}\sin\theta - \frac{3}{16}\,c\,H^2k^2\,\frac{\sinh 2kz}{\sinh^4 kD}\sin 2\theta \tag{74}$$

$$\frac{p^+}{\gamma} = \frac{H}{2}\,\frac{\cosh kz}{\cosh kD}\cos\theta + \frac{1}{8}\,H^2k\,[\,3\,\frac{\cosh 2kz}{\sinh 2kD} - 1\,]\,\frac{\cos 2\theta}{\sinh 2kD}$$

$$- \frac{1}{8}\,H^2k\,\frac{\cosh 2kz - 1}{\sinh 2kD} \tag{75}$$

The expression for u depends on the second order rotation. This is considered in chapter IX appendix II, in eqs. 54 and 55. For irrotational waves we have $\delta = 1/2$, and then u will be, using also eq. 61, 65, and 64,

$$u = c\,\frac{H}{2}\,k\,\frac{\cosh kz}{\sinh ky}\cos\theta + c(\frac{H}{2})^2\,\frac{k^2}{2}\coth kD\,\frac{\cosh kz}{\sinh ky}\cos 2\theta$$

$$+ c(\frac{H}{2})^2\,\frac{3}{2}k^2\,\frac{\coth kD}{\sinh^2 kD}\,\frac{\cosh 2kz}{\sinh 2ky}\cos 2\theta$$

$$+ c(\frac{H}{2})^2\,\frac{1}{2}k^2\coth kD\,[\,\frac{\cosh kz}{\sinh kD} - \frac{1}{kD}\,] \tag{76}$$

We get

$$\sinh ky = \sinh k(D+\eta) = \sinh kD\cosh k\eta + \cosh kD\sinh k\eta \tag{77}$$

The coefficient from the first term in eq. 76 can in a second order theory be approximated to

$$\frac{1}{\sinh ky} = \frac{1}{\sinh kD}\,\frac{1}{1+k\eta\,\coth kD} = \frac{1}{\sinh kD}\,(1-k\eta\,\coth kD) \tag{78}$$

Then we get from eq. 76

$$c\,\frac{H}{2}\,k\,\frac{\cosh kz}{\sinh ky}\cos\theta = c\,\frac{H}{2}\,k\,\frac{\cosh kz}{\sinh kD}\cos\theta - c\left(\frac{H}{2}\right)^2 k^2 \coth kD\,\frac{\cosh kz}{\sinh kD}\cos^2\theta \tag{79}$$

With

$$\cos^2\theta = \frac{1}{2}\,(\cos 2\theta + 1) \tag{8o}$$

We then see that eq. 76 and eq. 73 as wanted are identical within second order, using $y = D$ in second order terms.

w and p do not depend on the rotation of second order magnitude, so they are given by eqs. 62 and 63 with eq. 64. Using eq.78 we will again find that they within second order are identical with eqs. 74 and 75.

The standing wave could be considered in the same way. The standing wave of this chapter will be found to be irrotational. In the same way as for the progressive wave its rotations can be changed, if wanted.