CHAPTER V

STANDING FIRST ORDER WAVE AND WAVE PRESSURE

ABSTRACTS

In the preceding chapter we found the progressive first order wave. A different type of wave is the standing wave, which is important in connection with placing constructions. A vertical face breakwater will reflect incoming waves, and a system of standing waves will result. With incoming regular two dimensional waves and the vertical wall placed parallel to the wave fronts the resulting standing waves will be two-dimensional. We will consider those waves here. Solutions of different orders are given in the literature, but it is felt that even the first order solution can be made better. The classical first order solution is made by adding two identical progressive waves that travel against one another, the incoming wave and the reflected wave. This is a fast and logical way to get all necessary formulas.

We will here find the regular standing wave of first order by solving the basic hydrodynamic equations, like we did in chapter IV for the progressive wave. We will make the hydrodynamic problem more general and in this way we get both the progressive and the standing wave as solutions. The calculations here are just a little more complicated than for the progressive wave in chapter IV.

BASIC EQUATIONS

We consider two dimensional non stationary movement in incompressible frictionless water without surface tension. The bottom is horizontal.

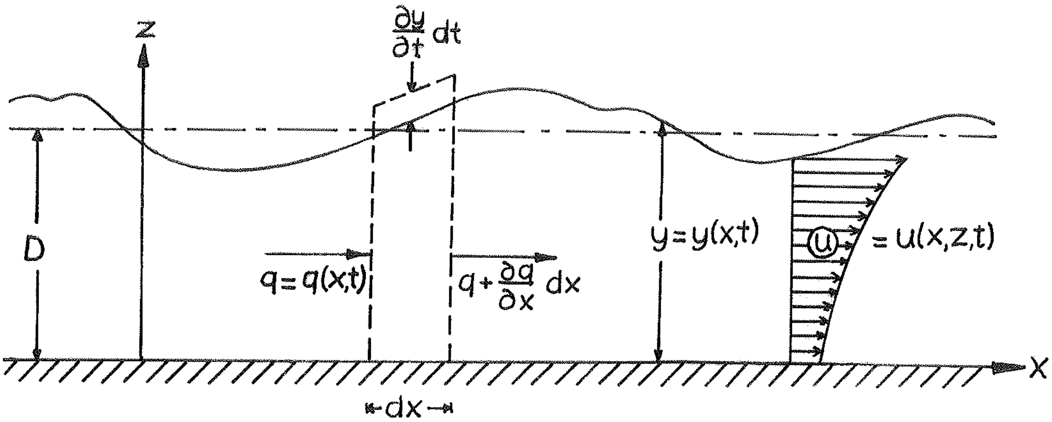

Fig. 1. Definition sketch.
It will be examined, what type of hydrodynamic motion we have when the vertical distribution of the horizontal velocity is hyperbolic cosine.

With the discharge $q = q(x,t)$ through a vertical, $y = y(x,t)$ the actual water depth, and $\eta = \eta(x,t)$ the surface elevation, the equation of continuity will give

$$\frac{\partial q}{\partial x} = - \frac{\partial y}{\partial t} = - \frac{\partial \eta}{\partial t} \qquad (1)$$

and we have

$$y = D + \eta \qquad (2)$$

where D is the mean depth.

The vertical coordinate is z with $z = o$ at the bottom. For the progressive wave of constant form we could integrate eq. 1 to find an expression for q (eq. 4 in chapter IV). But this is not possible this time.

The horizontal particle velocity $u = u(x,z,t)$ is integrated over a vertical to find q

$$q = \int_o^y u \, dz \tag{3}$$

Also this time we can expect u to have a cosh-distribution, so to fulfil eq. 3 we write

$$u = q \, R \, \frac{\cosh Rz}{\sinh Ry} \tag{4}$$

R is an unknown constant, which like before will be found to be $R = 2\pi/L$ in eq. 23. In a later chapter it will be shown how eq. 4 can be used in an arbitrary distribution of u . Here we only need to say : find the type of hydrodynamic problems that gives a u as given in eq. 4. It then 'happens so' that we rather easy get the standing and progressive wave solutions of eqs. 21 and 27.

u in eq. 4 is differentiated, using eq. 2

$$\frac{\partial u}{\partial x} = \frac{\partial q}{\partial x} R \frac{\cosh Rz}{\sinh Ry} - q \, R^2 \frac{\coth Ry \cosh Rz}{\sinh Ry} \frac{\partial \eta}{\partial x} \tag{5}$$

The local equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = o \tag{6}$$

gives the vertical particle velocity $w = w(x,z,t)$ by integration with the condition $w = o$ at the bottom $z = o$

$$w = \frac{\partial \eta}{\partial t} \frac{\sinh Rz}{\sinh Ry} + q \frac{\partial \eta}{\partial x} R \frac{\coth Ry \sinh Rz}{\sinh Ry} \tag{7}$$

In chapter IV we found the similar expression for the progressive wave in eq. 12. We then found that the last term in eq. 7 was small of second order compared to the first term, which is of first order.

Here eq. 7 is more general as it will also lead to the standing wave, but the same considerations can be made. This can best be done using the solutions found later in eqs. 21 and 26 or 27 and 3o.

So in a first order theory we can neglect the last term in eq. 7 and we get

$$w = \frac{\partial \eta}{\partial t} \frac{\sinh Rz}{\sinh Ry} \tag{8}$$

Together with eq. 4 this gives for the horizontal particle acceleration $G_x = G_x(x,z,t)$

$$G_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = \frac{\partial q}{\partial t} R \frac{\cosh Rz}{\sinh Ry} \tag{9}$$

The vertical particle acceleration $G_z = G_z(x,z,t)$ will be

$$G_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = \frac{\partial^2 \eta}{\partial t^2} \frac{\sinh Rz}{\sinh Ry} \tag{1o}$$

The vertical dynamic equation is

$$-\frac{\partial p}{\partial z} - \rho g = \rho G_z \tag{11}$$

where $p = p(x,z,t)$ is the pressure, g the acceleration of gravity, and ρ the unit mass of the water.

p is found by integration of eq. 11 with eq. 1o, and with the boundary condition $p = o$ at the surface $z = y$

$$\frac{p}{\gamma} = y - z + \frac{1}{g} \frac{\partial^2 \eta}{\partial t^2} \frac{1}{R} \frac{\cosh Ry - \cosh Rz}{\sinh Ry} \tag{12}$$

where γ is the unit weight, so $\gamma = \rho g$. By differentiation we find an expression for the horizontal pressure gradient

$$\frac{1}{\gamma} \frac{\partial p}{\partial x} = \frac{\partial \eta}{\partial x} + \frac{1}{g} \frac{\partial^3 \eta}{\partial x \partial t^2} \frac{1}{R} \frac{\cosh Ry - \cosh Rz}{\sinh Ry} \tag{13}$$

where we used eq. 2.

The horizontal dynamic equation

$$-\frac{\partial p}{\partial x} = \rho G_x \tag{14}$$

gives with eq. 9 another expression for $\partial p/\partial x$

$$\frac{1}{\gamma} \frac{\partial p}{\partial x} = - \frac{1}{g} \frac{\partial q}{\partial t} R \frac{\cosh Rz}{\sinh Ry} \tag{15}$$

Eqs. 13 and 15 are combined to give the first order wave equation

$$\frac{\partial \eta}{\partial x} + \frac{1}{g}\frac{\partial^3 \eta}{\partial x \partial t^2}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} + \frac{1}{g}\frac{\partial q}{\partial t} R \frac{\cosh Rz}{\sinh Ry} = o \qquad (16)$$

This equation is solved in the same way as explained in chapter II and IV. From eq. 16 we make a z-dependent equation

$$-\frac{1}{g}\frac{\partial^3 \eta}{\partial x \partial t^2}\frac{1}{R}\frac{\cosh Rz}{\sinh Ry} + \frac{1}{g}\frac{\partial q}{\partial t} R \frac{\cosh Rz}{\sinh Ry} = o \qquad (17)$$

and a z-independent equation

$$\frac{\partial \eta}{\partial x} + \frac{1}{g}\frac{\partial^3 \eta}{\partial x \partial t^2}\frac{1}{R}\coth RD = o \qquad (18)$$

Here we used that in a first order theory we can neglect η in

$$\coth Ry = \coth R(D + \eta) = \coth RD \qquad (19)$$

as shown in eq. 24 in chapter IV. Eq. 17 is differentiated with respect to x and eq. 1 is used

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \eta}{\partial t^2}\right) + \frac{\partial^2 \eta}{\partial t^2} R^2 = o \qquad (2o)$$

Eq. 2o gives an expression for $\partial^2\eta/\partial t^2$ and further for η containing a harmonic function and arbitrary functions.

Eq. 18 gives an expression for $\partial\eta/\partial x$ and further for η containing a harmonic function and an arbitrary function.

Comparing the two solutions for η we end up with only the harmonic solution.

STANDING WAVE SOLUTION

We get, as one possible solution

$$\eta = \frac{H}{2}\cos \omega t \cos Rx \qquad (21)$$

with

$$\omega = \sqrt{g R \tanh R D} \qquad (22)$$

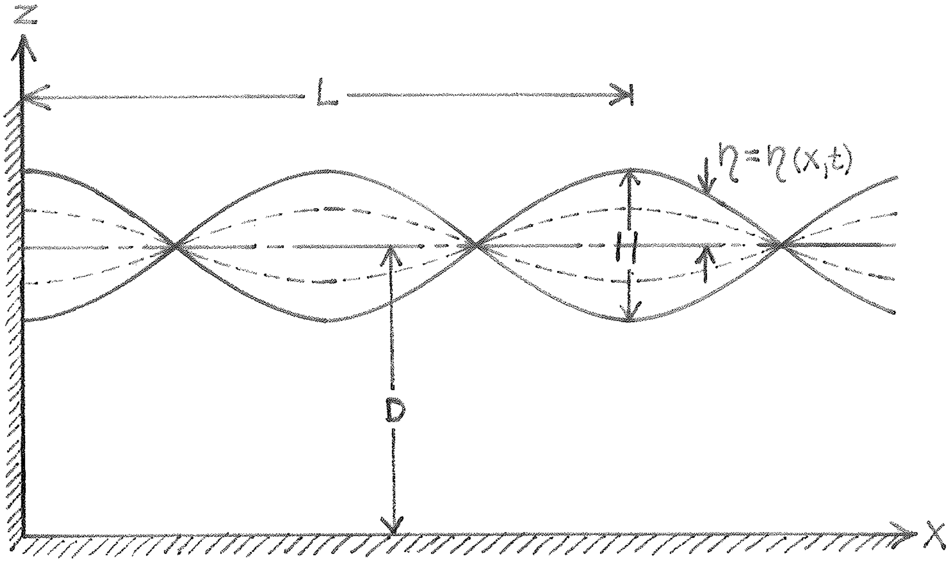

Fig. 2. Definition sketch for standing wave.

R is decided geometrically by eq. 21 to

$$R = \frac{2\pi}{L} = k \tag{23}$$

where L is the wave length.

In the same way ω is found to

$$\omega = \frac{2\pi}{T} \tag{24}$$

where T is the wave period. Eqs. 24, 23, and 22 give

$$\frac{\omega}{k} = \frac{L}{T} = \sqrt{\frac{g}{k} \tanh kD} \tag{25}$$

Eq. 1 then gives the solution for q

$$q = \frac{H}{2}\frac{\omega}{k} \sin \omega t \sin kx \tag{26}$$

with q = o at the vertical wall, x = o.

The solutions eqs. 21 - 26 give us the wanted regular standing wave of first order.

PROGRESSIVE WAVE SOLUTION

From eqs. 2o and 18 we can also get the solution for the progressive wave

$$\eta = \frac{H}{2} \cos R(x - ct) \tag{27}$$

with

$$R = \frac{2\pi}{L} = k \tag{28}$$

$$c = \frac{L}{T} = \sqrt{\frac{g}{k} \tanh kD} \tag{29}$$

$$q = c\,\eta = \frac{H}{2}\,\frac{L}{T} \cos k(x - ct) \tag{3o}$$

The progressive wave was considered in chapter IV so here we concentrate on the standing wave.

FORMULAS OF THE FIRST ORDER STANDING WAVE

$$\eta = \frac{H}{2} \cos \omega t \cos kx \tag{31}$$

$$k = \frac{2\pi}{L} \tag{32}$$

$$\omega = \frac{2\pi}{T} \tag{33}$$

$$\frac{\omega}{k} = \frac{L}{T} = \sqrt{\frac{g}{k} \tanh kD} \tag{34}$$

$$q = \frac{H}{2} \frac{L}{T} \sin \omega t \sin kx \tag{35}$$

$$u = q\, k \frac{\cosh kz}{\sinh ky} \tag{36}$$

$$w = \frac{\partial \eta}{\partial t} \frac{\sinh kz}{\sinh ky} \tag{37}$$

$$\frac{\partial \eta}{\partial t} = - \frac{H}{2} \omega \sin \omega t \cos kx \tag{38}$$

$$\frac{p}{\gamma} = y - z + \frac{1}{g} \frac{\partial^2 \eta}{\partial t^2} \frac{1}{k} \frac{\cosh ky - \cosh kz}{\sinh ky} \tag{39}$$

from eq. 31 we get

$$\frac{\partial^2 \eta}{\partial t^2} = -\omega^2 \eta \tag{40}$$

using eq. 34 the pressure of eq. 39 can be written

$$\frac{p}{\gamma} = y - z - \eta \frac{\tanh kD}{\tanh ky} \left[1 - \frac{\cosh kz}{\cosh ky} \right] \tag{41}$$

Eq. 10 gives the vertical acceleration at the surface $z = y$

$$G_{zs} = \frac{\partial^2 \eta}{\partial t^2} \tag{42}$$

Then eq. 39 can be written

$$\frac{p}{\gamma} = y - z + \frac{G_{zs}}{g} \frac{1}{R} \frac{\cos Ry - \cosh Rz}{\sinh Ry} \tag{43}$$

Eqs. 41 and 43 for the pressure of the standing wave are seen to be the same as eqs. 38 and 48 of chapter IV for the pressure of the progressive wave.

Eq. 93 in chapter IV can also be used for the standing wave, under the same considerations as made then.

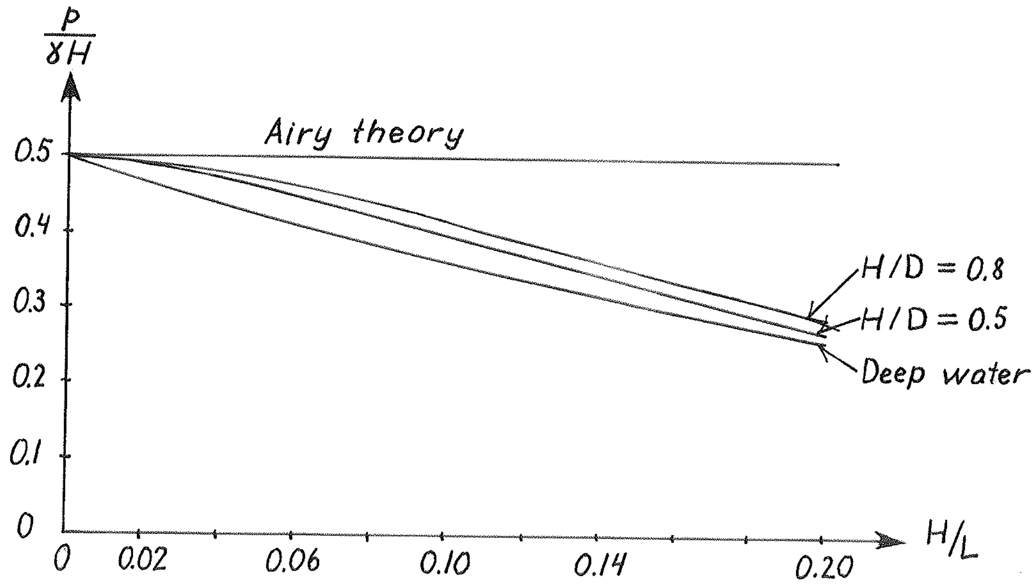

Fig. 3. The pressure on the wall, below the maximum crest, at the mean water level where the maximum wave pressure is exerted. Comparison of the first order theory of this chapter with the Airy theory. It is seen that the vertical acceleration can give a substantiel reduction of the hydrostatic pressure. This maximum pressure will in most cases only be changed slightly by higher order calculations.

DEEP WATER STANDING WAVE

In the case of infinite deep water, $D \to \infty$, or rather $D/L \to \infty$, we get more simple expressions. The progressive deep water wave was found independently in chapter II to illustrate the method of finding waves. But we also saw in chapter IV how to get the same deep water wave from the wave on water of arbitrary depth. Eqs. 31, 32, 33, 35, and 38 are unchanged. The others will be changed.

$$\frac{D}{L} \to \infty \qquad (44)$$

$$kD \to \infty \quad \text{and} \quad ky \to \infty \qquad (45)$$

$$\tanh kD \to 1 \qquad (47)$$

$$\frac{L}{T} \to \sqrt{\frac{g}{k}} \qquad (47)$$

$$\sinh ky \to \frac{1}{2} e^{ky} \qquad (48)$$

$$\sinh kz \to \frac{1}{2} e^{kz} \qquad (49)$$

$$u \to q\, k\, e^{k(z - y)} \qquad (5o)$$

$$w \to \frac{\partial \eta}{\partial t} e^{k(z - y)} \qquad (51)$$

$$\frac{p}{\gamma} \to D - z + \eta e^{k(z - y)} \qquad (52)$$

In eqs. 5o, 51. and 52 z and y are measured from the bottom. But y -z is the distance below the surface and D - z the distance below the mean water level.

If we use a coordinate system like for the progressive deep water wave of chapter II with $z = o$ at the mean water level we get for u, w, and p

$$u = q\,k\,e^{k(z - \eta)} \tag{53}$$

$$w = \frac{\partial \eta}{\partial t}\, e^{k(z - \eta)} \tag{54}$$

$$\frac{p}{\gamma} = -z + \eta\, e^{k(z - \eta)} \tag{55}$$

It is seen that these expressions can also be used for the progressive wave.

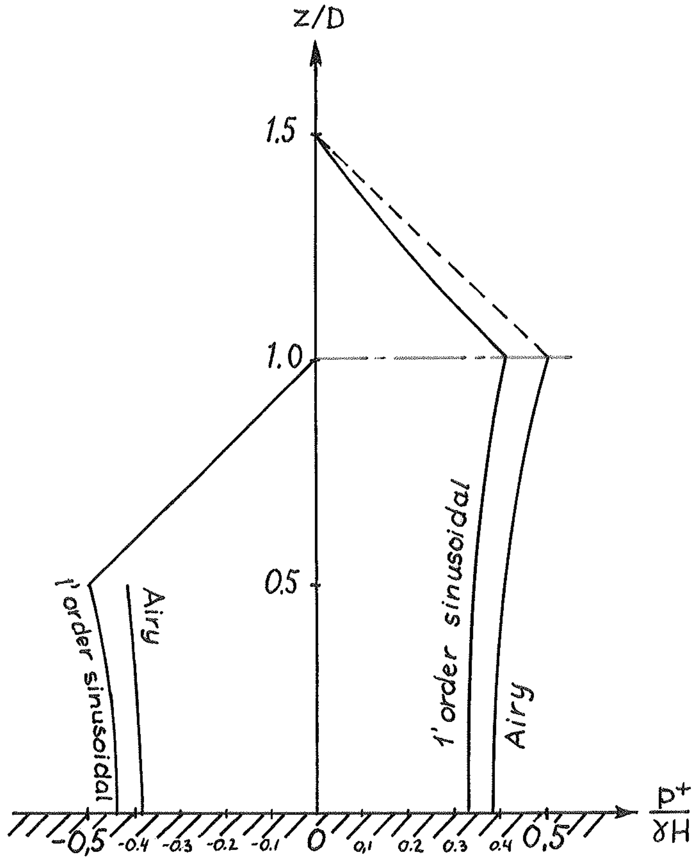

Fig. 4. Maximum and minimum wave pressure, $p^+/(\gamma H)$, at the vertical wall in a standing wave with the steepness $H/L = 12\%$ and $H/D = 1.o$. The Airy theory gives the hydrostatic pressure at the mean water level, so the shown dotted line can as well be used for $z > o$. A similar proposal is more difficult to use for the trough because the pressure at the surface will not be $p = o$.

SHALLOW WATER STANDING WAVE

Eqs. 31, 32, 33, 35, and 38 are unchanged. The others are changed.

$$\frac{L}{D} \rightarrow \infty \quad \text{or} \quad \frac{D}{L} \rightarrow 0 \tag{56}$$

$$\tanh kD \rightarrow kD = 2\pi \frac{D}{L} \tag{57}$$

$$\frac{L}{T} \rightarrow \sqrt{g\ D} \tag{58}$$

$$\sinh ky \rightarrow ky \tag{59}$$

$$\cosh ky \rightarrow 1 \tag{60}$$

$$u \rightarrow \frac{q}{y} \tag{61}$$

$$w \rightarrow \frac{\partial \eta}{\partial t} \frac{z}{y} \tag{62}$$

$$\frac{p}{\gamma} \rightarrow y - z \tag{63}$$

Eq. 43 will be

$$\frac{p}{\gamma} \rightarrow y - z + \frac{G_{zs}}{g} \frac{y^2 - z^2}{2y} \tag{64}$$

WAVE PRESSURE

A very important problem for the engineer is to decide the wave pressure exerted upon structures. In many cases the design pressure for the geotechnical stability of the structure can be decided by the maximum pressure from the standing wave. The design negative pressure (the wave suction) is also decided from the standing wave. (Disasters have occured, where vertical face breakwaters have been sucked out into the ocean. Experiments, (and the formulas of this chapter) show that the 'sucking' force can be bigger than the positive pressure force.) For the stability we have an interest in the integrated forces, the total horizontal wave force P^+ and the overturning moment around the foot point M^+.

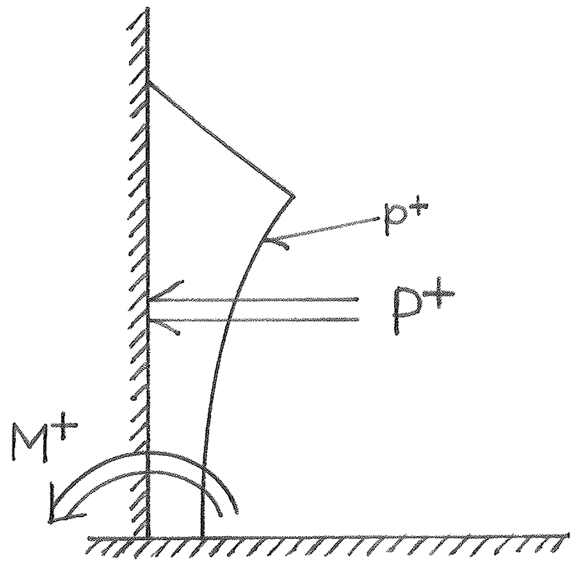

Fig. 5. The pressure on a vertical wall. P^+ and M^+ are integrated results of p^+. P^+ is the total horizontal (sliding) force and M^+ is the overturning moment around the footpoint. P^+ and M^+ are needed for investigating the geotechnical stability.

From eq. 43 we find the total horizontal force P

$$\frac{P}{\gamma} = \int_0^y \frac{p}{\gamma}\, dz = \int_0^y [y - z + \frac{G_{zs}}{g}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry}]\, dz$$

$$= \frac{y^2}{2} + \frac{G_{zs}}{g}\frac{1}{R}[y \coth Ry - \frac{1}{R}] \qquad (65)$$

For the simple first order wave we use eqs. 42 and 4o for G_{zs} like we did in eq. 41 and we get

$$\frac{P}{\gamma} = \frac{y^2}{2} - \eta \tanh kD\, [y \coth ky - \frac{1}{k}] \qquad (66)$$

The overturning moment around the footpoint of the vertical wall is found from eq. 43 by

$$\frac{M}{\gamma} = \int_0^y \frac{p}{\gamma}\, z\, dz = \int_0^y [y - z + \frac{G_{zs}}{g}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry}]\, z\, dz$$

$$= \frac{y^3}{6} + \frac{G_{zs}}{g}\frac{1}{R}\left\{ [\frac{y^2}{2} + \frac{1}{R^2}] \coth Ry - \frac{y}{R} - \frac{1}{R^2 \sinh Ry}\right\} \qquad (67)$$

Like before we can get a more simple expression by using eqs. 42 and 4o for G_{zs}.

$$\frac{M}{\gamma} = \frac{y^3}{6} - \eta \tanh kD \left\{ [\frac{y^2}{2} + \frac{1}{k^2}] \coth ky - \frac{y}{k} - \frac{1}{k^2 \sinh ky}\right\} \qquad (68)$$

The wave presure is defined as the difference between the water pressure below the wave and the hydrostatic pressure from the mean water level

$$\frac{p^+}{\gamma} = \frac{p}{\gamma} - (D - z) \qquad (69)$$

We then get for the total wave force

$$\frac{P^+}{\gamma} = \frac{P}{\gamma} - \frac{D^2}{2} \qquad (7o)$$

and for the overturning wave moment around the footpoint

$$\frac{M^+}{\gamma} = \frac{M}{\gamma} - \frac{D^3}{6} \qquad (71)$$

The wave of this chapter is the same as the classical first order wave (the standing Airy wave) within first order approximations. So the differences are of second order. Those differences should be negligible according to the theory but experiments show them to be very important.

When an engineer is faced with the problem of finding the wave pressure on a vertical wall it is of interest to consider the movement of the water close to the wall. Using the Airy theory he will quickly be faced with problems of contradictions.

The water moves vertically up and down with the surface following the Airy expression

$$\eta = \frac{H}{2} \cos \omega t \tag{72}$$

This gives a vertical velocity of the surface particle of

$$w_s = \frac{\partial \eta}{\partial t} = - \frac{H}{2} \omega \sin \omega t \tag{73}$$

while the Airy formula for the vertical velocity would give

$$w_s = - \frac{H}{2} \omega \sin \omega t \, \frac{\sin k(D + H/2)}{\sin kD} \tag{74}$$

As H for the standing wave in practical problems easily can be as big as D it is seen that there can be a rather big numerical difference between eqs. 73 and 74.

This difference is bigger and more important for the vertical acceleration.

But the most important difference is found for the pressure. The Airy expression for the wave pressure is

$$\frac{p^+}{\gamma} = \eta \frac{\cosh kz}{\cosh kD} \tag{75}$$

This expression can though not be used above the mean water level so e.g. at the Technical University of Denmark it is proposed to use the hydrostatic pressure for $z \geq D$

$$\frac{p^+}{\gamma} = y - z \tag{76}$$

This gives though a rather high positive pressure for steep waves as it can be seen in chapter III on fig. 6.

At the same time the negative pressure or the 'wave sucking' can be too small so it is questionable for the practical engineer to use this proposal. Eq. 76 has the advantage compared to other more complicated expressions from the literature that it gives the pressure $p^{+} = o$ at the surface. But to give an expression as eq. 76 it is necessary that the wave has got no vertical acceleration above the mean water level. And here the acceleration is of significant importance for waves of practical interest.

Instead eq. 41 for the pressure in first order waves should be used. It is felt to be a reasonable simple expression to use, specially with the hand computers in common use.

Eqs. 66 and 68 have been compared to modeltests performed by the author as a student in 1968. Those modeltests have been of decisive importance for the author's understanding of wave motion at the vertical wall breakwater and resulted a year later in the equations of this chapter, so modeltests and observations of nature are felt to be important steps in making theoretical work.

It is seen that there is a reasonable good agreement between experiments and theory, except for the negative pressure, the 'wave sucking', below trough of the less steep waves, where the sucking force is estimated too high. This is because we have used the first order theory all the way through so that we let both the crest and the trough be $H/2$ high and deep. But for higher waves on shallow waters the trough is significant less than $H/2$ deep giving a 'sucking force' less than that of eqs. 66 and 68. This disadvantage can be overcome by using a more realistic η in eqs. 65 and 67 and if possible estimate realistic G_{zs} and R.

Within the scope of a first order theory there is a big freedom in estimating, without going beyond the hydrodynamic limits of the theory.

The positive pressure from the crest gives better agreement between theory and experiments. This could also be expected. The crest is higher than the $H/2$ used in the pressure equations. But this does not make the actual pressure bigger than the calculated, because at the same time the actual negative vertical acceleration is bigger than the calculated, which reduces the pressure from the higher crest. In this way we end up with close to the right pressure in our calculations. The pressure reducing effect from the bigger negative acceleration of the higher crest can also be seen on the pressure at the mean water level, the level where we have the biggest pressures.

In chapter XII the pressure from the second order cnoidal deep water wave is compared to the first order pressure, and it is seen how well the first order expression of this chapter predicts the maximum pressure. In the same chapter it is seen that the effect of the negative vertical acceleration of the water at the time of crest by the wall can be so big that the wave pressure even can become negative at greater depths. Such effects of double humps on the time-pressure curve cannot be explained by the first order theory of eq. 41. It can be explained by eq. 43 though, with proper estimation of η, G_{zs} and R.

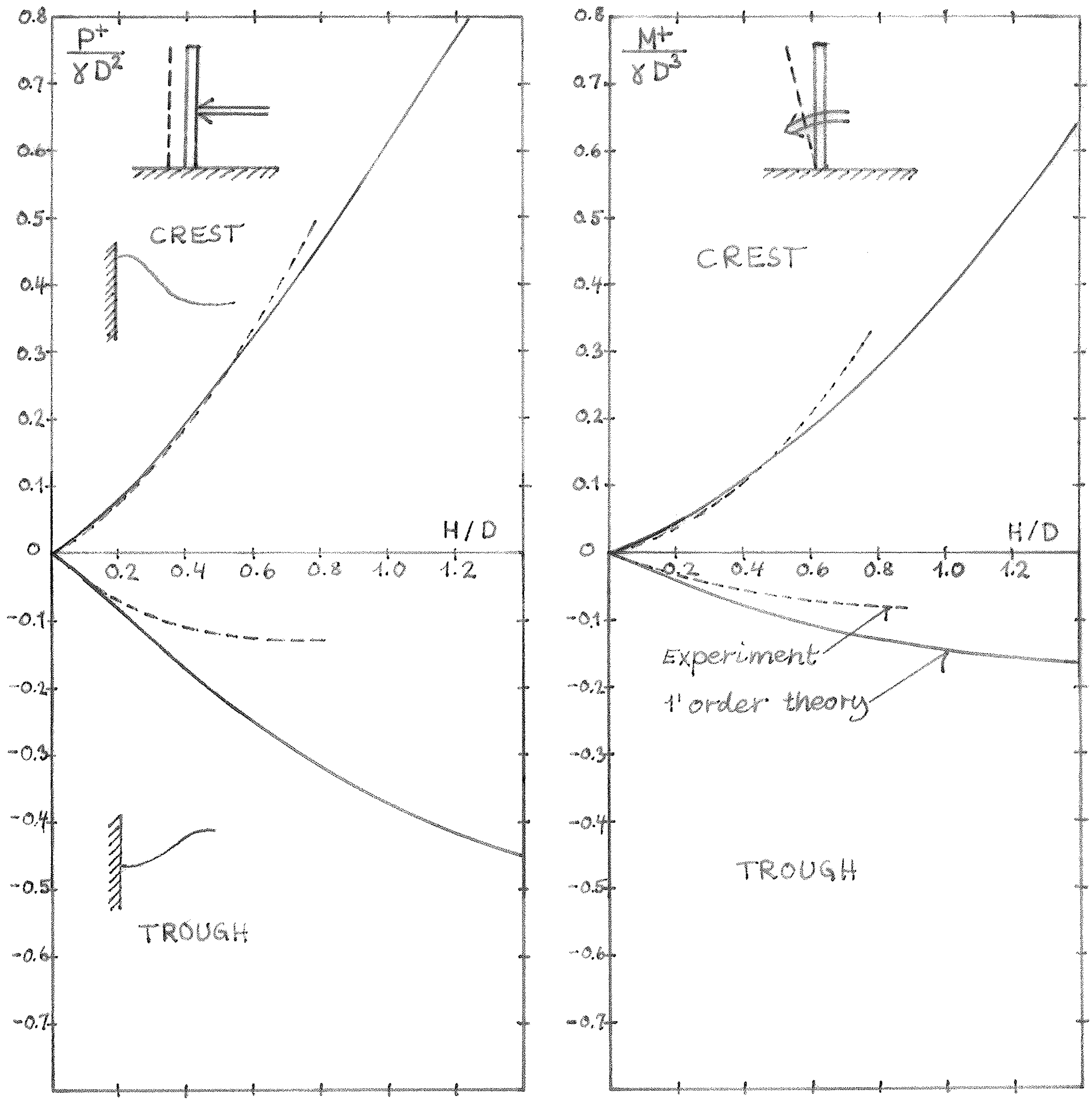

Fig. 6. H/L = 3% (where H is the wave height of the standing wave). The total horizontal (sliding) force and the overturning moment around the footpoint from the wave pressure on a vertical wall when it is maximum and when it is minimum. Comparison of the first order sinusoidal theory of this chapter with model tests.

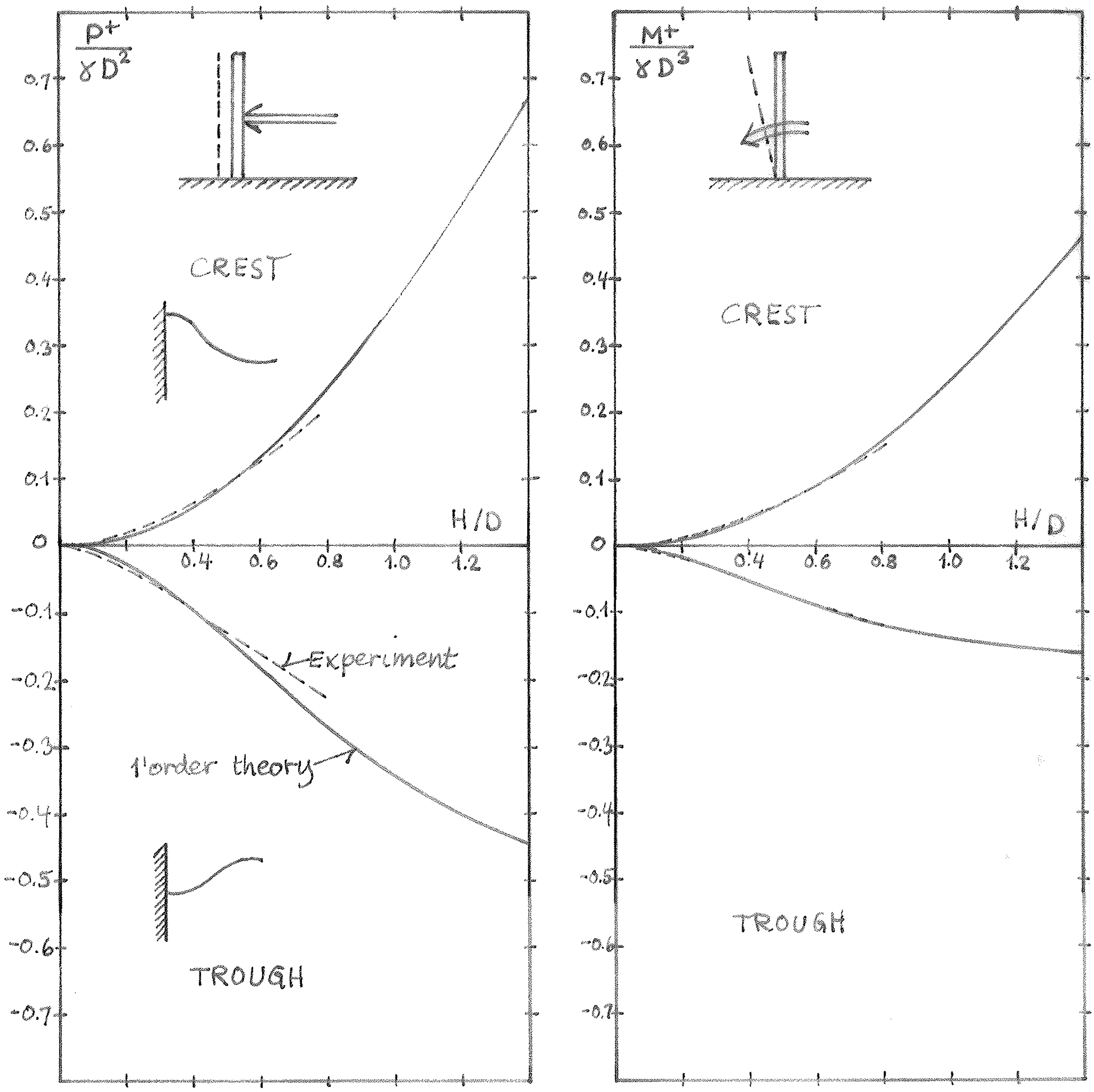

Fig. 7. H/L = 16% .
For the relevant situation with incoming design waves of a wave height of around 7 m and a wave period of 1o sec, the wave steepness for the standing wave will be around 16% for a vertical face breakwater placed at a depth of 1o m.