CHAPTER IV

PROGRESSIVE FIRST ORDER WAVE ON ARBITRARY DEPTH

ABSTRACTS

The first order sinusoidal wave on arbitrary depth will here be found in the same way as used in chapter II for the deep water wave. The wave solution is the same as the wellknown Airy wave, except that the expressions for velocities and pressure in a natural way come out with a second order difference that makes them fit the boundary conditions better. The expressions here are longer than for the deep water wave, but the principles are the same, so the explanations are shorter this time.

BASIC EQUATIONS

We consider a two dimensional progressive wave of permanent form on incompressible frictionless water without surface tension, and the bottom is horizontal.

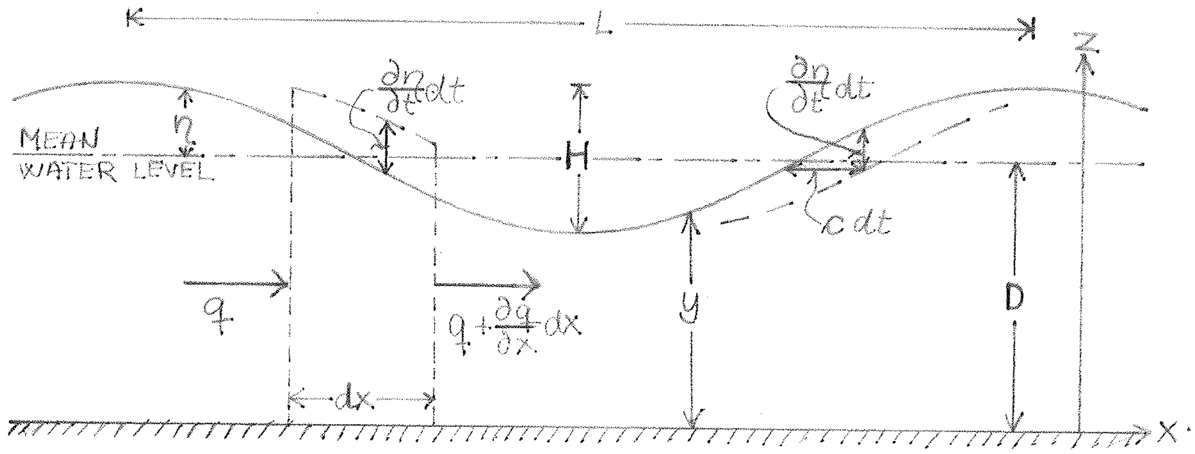

Fig. 1. Definition sketch.

Through a vertical we have the discharge $q = q(x,t)$, so the equation of continuity will give the wellknown expression for the surface elevation $\eta = \eta(x,t)$

$$\frac{\partial q}{\partial x} = -\frac{\partial y}{\partial t} = -\frac{\partial \eta}{\partial t} \tag{1}$$

y is the actual water depth, so with D being the mean depth we have

$$y = D + \eta \tag{2}$$

It is convenient to have an expression for the actual (variable) water depth, and in open channel hydraulics in Denmark the letter y is used. It must not be confused with the vertical coordinate z.

z is measured from the horizontal bottom.

For a permanent wave we have

$$-\frac{\partial \eta}{\partial t} = c\frac{\partial \eta}{\partial x} \tag{3}$$

where c is the wave celerity. Eqs. 1 and 3 give for q in a progressive wave without a resultant discharge

$$q = c\eta \tag{4}$$

η is measured from the mean water level so

$$\int_0^L \eta \, dx = 0 \tag{5}$$

or

$$\int_0^T \eta \, dt = 0 \tag{6}$$

L is the wave length and T the wave period so

$$L = c\,T \tag{7}$$

Besides from eq. 4, q can be found by the integration of the horizontal particle velocity $u = u(x,z,t)$ over a vertical from the bottom to the surface

$$q = \int_0^y u \, dz \tag{8}$$

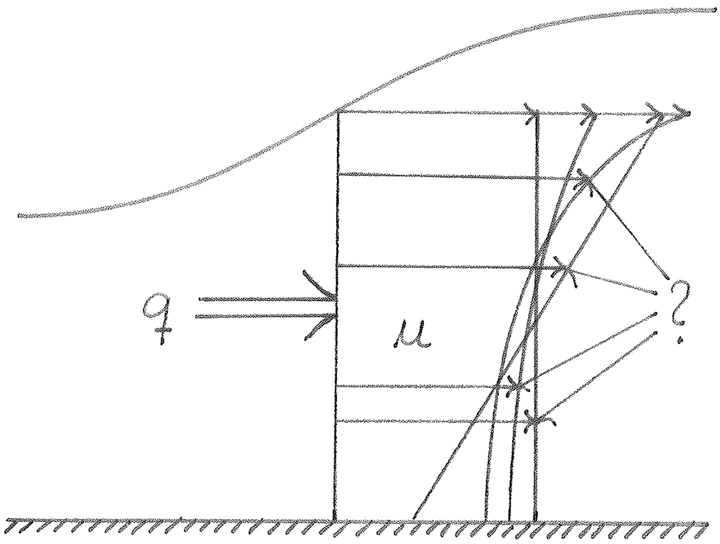

Fig. 2. The result of the integration of u over a vertical is fixed, but the vertical distribution of u is unknown. In this chapter the distribution is assumed to be cosh, but in chapter VII it is written as an arbitrary function.

This time we expect u to have a cosh - distribution (like the Airy wave). The possibility of finding a wave solution later in eq. 28 also shows this to be correct. In chapter VII u will be chosen with an arbitrary distribution. So to fulfil eq. 8, u will be written

$$u = q\,R\,\frac{\cosh Rz}{\sinh Ry} = c\,\eta\,R\,\frac{\cosh Rz}{\sinh Ry} \qquad (9)$$

Here eq. 4 was used. R is an unknown constant, which will be found to be $R = 2\pi/L$ in eq. 29.

u is differentiated to give

$$\frac{\partial u}{\partial x} = c\frac{\partial \eta}{\partial x} R \frac{\cosh Rz}{\sinh Ry} - c\eta R^2 \frac{\coth Ry \cosh Rz}{\sinh Ry}\frac{\partial \eta}{\partial x} \qquad (10)$$

Here eq. 2 was used in the last term.

With the local equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = o \qquad (11)$$

the vertical particle velocity $w = w(x,z,t)$ is found by integration with the boundary condition that at the bottom $z = o$ we have $w = o$

$$w = - c \frac{\partial \eta}{\partial x} \frac{\sinh Rz}{\sinh Ry} [1 - \eta R \coth Ry] \qquad (12)$$

In a first order theory the last term can be neglected and we get

$$w = - c \frac{\partial \eta}{\partial x} \frac{\sinh Rz}{\sinh Ry} \qquad (13)$$

This demands that

$$\eta R \coth Ry \ll 1 \qquad (14)$$

The situation here is more complicated than for the deep water wave. We must still have that ηR is small, which means that the wave steepness H/L must be small. At the same time coth Ry must not become too big, which means that Ry (or the water depth) must not be too small. The condition in eq. 14 is considered again later in eq. 58.

From eqs. 9 and 13 we get the first order approximation of the horizontal particle acceleration $G_x = G_x(x,z,t)$ using eq. 3

$$G_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = - c^2 \frac{\partial \eta}{\partial x} R \frac{\cosh Rz}{\sinh Ry} \qquad (15)$$

The vertical particle acceleration $G_z = G_z(x,z,t)$ will be

$$G_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = c^2 \frac{\partial^2 \eta}{\partial x^2} \frac{\sinh Rz}{\sinh Ry} \qquad (16)$$

The vertical dynamic equation is

$$-\frac{\partial p}{\partial z} - \rho g = \rho G_z \qquad (17)$$

where $p = p(x,z,t)$ is the pressure, g the acceleration of gravity, and ρ the unit mass of the water

From eqs. 16 and 17 p is found by integration with the boundary condition that at the surface z = y there is no water pressure, p = o

$$\frac{p}{\gamma} = y - z + \frac{c^2}{g}\frac{\partial^2\eta}{\partial x^2}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} \tag{18}$$

where γ is the unit weight, so $\gamma = \rho g$.

By differentiation and using eq. 2 the horizontal pressure gradient is found to

$$\frac{1}{\gamma}\frac{\partial p}{\partial x} = \frac{\partial\eta}{\partial x} + \frac{c^2}{g}\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} \tag{19}$$

The horizontal dynamic equation is

$$-\frac{\partial p}{\partial x} = \rho G_x \tag{20}$$

Using eq. 15 this gives another expression for $\partial p/\partial x$

$$\frac{1}{\gamma}\frac{\partial p}{\partial x} = \frac{c^2}{g}\frac{\partial\eta}{\partial x}R\frac{\cosh Rz}{\sinh Ry} \tag{21}$$

eqs. 19 and 21 are combined to give the first order wave equation

$$\frac{\partial\eta}{\partial x} + \frac{c^2}{g}\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} - \frac{c^2}{g}\frac{\partial\eta}{\partial x}R\frac{\cosh Rz}{\sinh Ry} = 0 \tag{22}$$

WAVE SOLUTION

Eq. 22 must be fulfilled for all z , also for the surface z = y

$$\frac{\partial\eta}{\partial x} - \frac{c^2}{g}\frac{\partial\eta}{\partial x}R\coth Ry = 0 \tag{23}$$

By the use of eq. 2 coth Ry can be expanded as

$$\coth Ry = \coth R(D+\eta) = \frac{\coth RD + \tanh R\eta}{1 + \coth RD \tanh R\eta}$$

$$= (\coth RD + \tanh R\eta)(1 - \coth RD \tanh R\eta) + \cdots$$

$$= \coth RD - [\coth^2 RD - 1]\tanh R\eta + \cdots$$

$$= \coth RD - \frac{R\eta}{\sinh^2 RD} + \cdots \tag{24}$$

So in a first order theory coth Ry in eq. 23 can be substituted by coth RD.

The wave celerity is then found from eq. 23

$$c = \sqrt{\frac{g}{R} \tanh RD} \tag{25}$$

Eq. 22 is solved by splitting it into two equations, an equation of the z-dependent terms and an equation not depending on z. As discussed for the deep water wave in chapter II it is only necessary to consider the z-dependent equation, now that we have solved the surface equation, eq. 23. The z-dependent equation from eq. 22 gives

$$\frac{c^2}{g} \frac{\partial^3 \eta}{\partial x^3} \frac{1}{R} \frac{\cosh Rz}{\sinh Ry} + \frac{c^2}{g} \frac{\partial \eta}{\partial x} R \frac{\cosh Rz}{\sinh Ry} = 0 \tag{26}$$

or

$$\frac{\partial^3 \eta}{\partial x^3} + \frac{\partial \eta}{\partial x} R^2 = 0 \tag{27}$$

This leads to the solution for a progressive wave

$$\eta = \frac{H}{2} \cos R(x - ct) \tag{28}$$

so that R must be

$$R = \frac{2\pi}{L} = k \tag{29}$$

So the solutions of eqs. 25, 28, and 29 are the same as for the Airy wave.

The wave will be found to be irrotational.

FORMULAS OF THE FIRST ORDER PROGRESSIVE WAVE

We will now review the most important formulas

$$\eta = \frac{H}{2}\cos k(x - ct) \tag{3o}$$

$$k = \frac{2\pi}{L} \tag{31}$$

$$c = \frac{L}{T} = \sqrt{\frac{gL}{2\pi}\tanh kD} = \sqrt{\frac{g}{k}\tanh kD} \tag{32}$$

$$u = c\eta k\frac{\cosh kz}{\sinh ky} \tag{33}$$

$$w = -c\frac{\partial\eta}{\partial x}\frac{\sinh kz}{\sinh ky} = \frac{\partial\eta}{\partial t}\frac{\sinh kz}{\sinh ky} \tag{34}$$

$$\frac{\partial\eta}{\partial x} = -\frac{H}{2}k\sin k(x - ct) = -\frac{1}{c}\frac{\partial\eta}{\partial t} \tag{35}$$

$$\frac{p}{\gamma} = y - z + \frac{c^2}{g}\frac{\partial^2\eta}{\partial x^2}\frac{1}{k}\frac{\cosh ky - \cosh kz}{\sinh ky} \tag{36}$$

From eq.3o we get

$$\frac{\partial^2\eta}{\partial x^2} = -k^2\eta \tag{37}$$

Using eq. 32 the pressure in eq. 36 can then be written

$$\frac{p}{\gamma} = y - z - \eta\frac{\tanh kD}{\tanh ky}\left[1 - \frac{\cosh kz}{\cosh ky}\right] \tag{38}$$

Eq. 36 can also be changed in a different way. The vertical acceleration at the surface $z = y$ will be, using eq. 16

$$G_{zs} = c^2\frac{\partial^2\eta}{\partial x^2} \tag{39}$$

Then eq. 36 will be

$$\frac{p}{\gamma} = y - z + \frac{G_{zs}}{g}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} \tag{4o}$$

This expression can be of practical use for e.g. irregular waves when G_{zs} and R can be estimated in a reasonable way.

DEEP WATER AND SHALLOW WATER LIMIT

For infinite deep water $D \to \infty$ we have

$$\frac{D}{L} \to \infty \tag{41}$$

$$kD \to \infty \quad \text{and} \quad ky \to \infty \tag{42}$$

$$\tanh kD \to 1 \tag{43}$$

$$c \to \sqrt{\frac{gL}{2\pi}} = \sqrt{\frac{g}{k}} \tag{44}$$

$$\sinh ky \to \frac{1}{2}e^{ky} \tag{45}$$

$$\cosh kz \to \frac{1}{2}e^{kz} \tag{46}$$

$$u \to c\eta k\, e^{k(z-y)} \tag{47}$$

Eq. 44 for c shows as wanted the same expression as eq. 4o in chapter II. In eq. 47 z and y are both measured from the bottom at infinite depth. But the difference $z - y$ is the negative distance below the surface of the point considered, just like $z - \eta$ in eq. 42 in chapter II. So the two expressions for u are the same. In the same way eqs. 34 and 36 for w and p will coincide with eqs. 43 and 44 in chapter II for $D \to \infty$.

SHALLOW WATER PROGRESSIVE WAVE

Eqs. 3o, 31, and 35 are unchanged. The others are changed.

$$\frac{L}{D} \rightarrow \infty \quad \text{or} \quad \frac{D}{L} \rightarrow 0 \tag{48}$$

$$\tanh kD \rightarrow kD = 2\pi \frac{D}{L} \tag{49}$$

$$c \rightarrow \sqrt{g D} \tag{5o}$$

$$\sinh ky \rightarrow ky \tag{51}$$

$$\cosh ky \rightarrow 1 \tag{52}$$

$$u \rightarrow c \frac{\eta}{y} \tag{53}$$

$$w \rightarrow \frac{\partial \eta}{\partial t} \frac{z}{y} \tag{54}$$

$$\frac{p}{\gamma} \rightarrow y - z \tag{55}$$

Eq. 4o will be

$$\frac{p}{\gamma} \rightarrow y - z + \frac{G_{zs}}{g} \frac{y^2 - z^2}{2y} \tag{56}$$

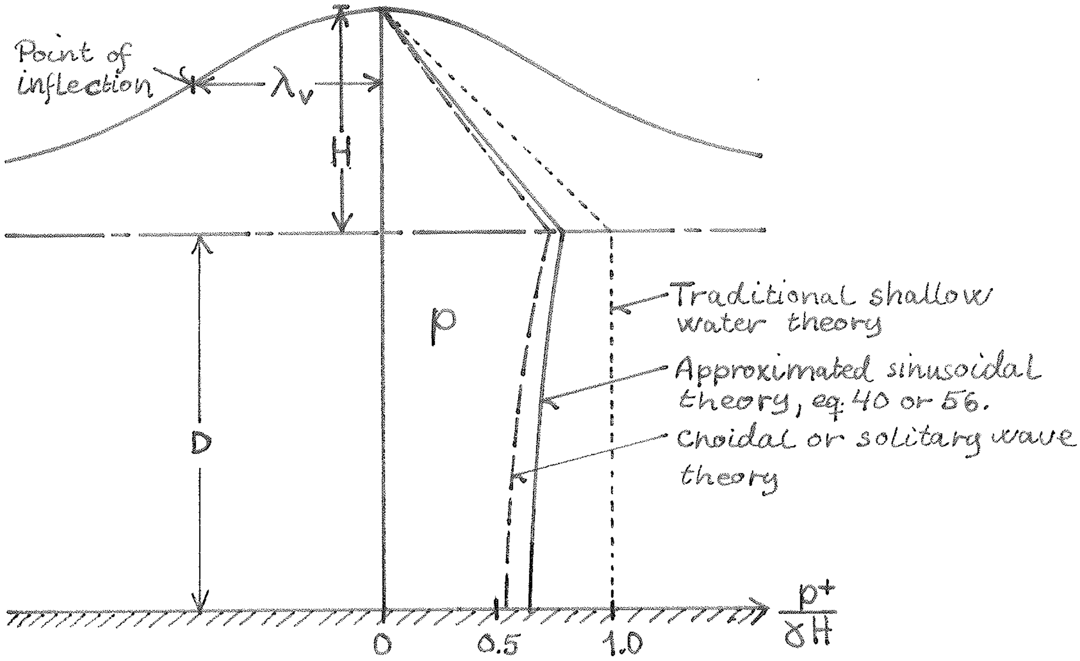

Fig. 3. Pressure below the solitary wave, H/D = o.6, according to different theories. The wave profile is drawn with a horizontal scale that is half the vertical scale. The traditional shallow water theory simply gives the hydrostatic pressure, neglecting the influence of the vertical acceleration.
In chapter VIII an expression for the pressure in cnoidal and solitary waves is given, which is also shown here.
But the simple expressions from this chapter can also be used. In eqs. 4o and 56 the surface acceleration G_{zs} and the distribution factor R should be estimated in a reasonable and practical way. G_{zs} is found from eqs. 39 and 37 substituting k with R. For R we have here used $R = 2\pi/(12\lambda_v)$, because of the following considerations :
In the sinusoidal wave L/4 is the horizontal distance from the crest to the point with η = o. For the solitary wave, which has the point of inflection for $\eta/H = 2/3$, we then say that the η = o point in a sinusoidal approximation is 3 times further down from the crest than the point of inflection, so it is estimated also to be 3 times further out, i.e. $L_{sin}/4 = 3\lambda_v$.
The result obtained is seen to be reasonable. Eqs. 4o and 56 give very close to the same graph.
There are other reasonable considerations, than the one used above.

CONDITION ON WAVE HEIGHT

We will now consider the condition in eq. 14 again. For shallow water waves we have

$$\eta R \coth Ry \to \frac{\eta}{y} \quad \text{for} \quad \frac{D}{L} \to 0 \tag{57}$$

This means that we in eq. 14 must demand the relative wave height to be small

$$\frac{H}{D} \ll 1 \qquad \text{for} \quad \frac{D}{L} \to 0 \tag{58}$$

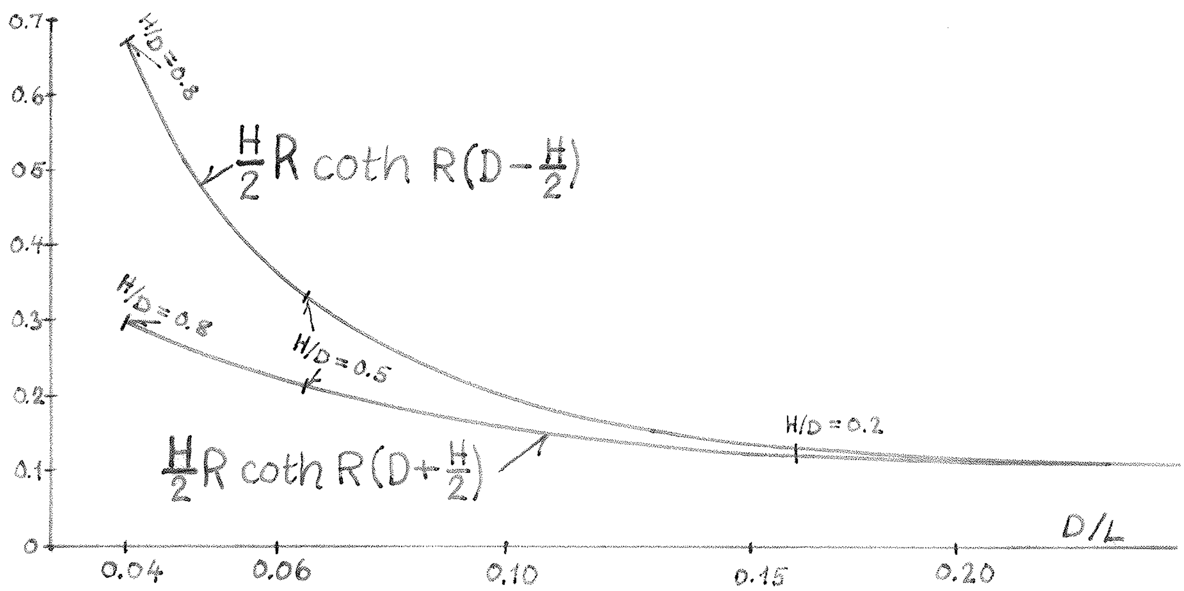

Fig. 4. The second order term in the vertical velocity for the wave with H/L = 3.2%.
$\eta R \coth Ry$ should be small compared to 1. This term is shown above for $\eta = H/2$ (crest) and $\eta = -H/2$ (trough) and with $R = k$.
Even for the trough of a wave with $H/D = 0.8$ the second order term here does not exceed the first order term.

APPENDIX IV

AIRY EXPRESSIONS

From the classical Airy wave we have

$$\frac{p^+}{\gamma} = \eta \frac{\cosh kz}{\cosh kD} \tag{59}$$

where p^+ is the wave pressure, i.e. water pressure above hydrostatic pressure from the mean water, so

$$p^+ = p - \gamma(D - z) \tag{60}$$

Eq. 59 will result from eq. 38 with further first order approximations.

Approximating ky with kD eqs. 58 and 38 will give

$$\frac{p^+}{\gamma} = \eta - \eta[1 - \frac{\cosh kz}{\cosh kD}] = \eta \frac{\cosh kz}{\cosh kD} \tag{61}$$

the same as eq. 59.

The Airy expression eq. 59 has the practical problem, that it can not be used above mean water level $z > D$. This is not a problem with eqs.36, 38, and 40. They also give the exact pressure at the surface, $p = 0$ for $z = y$.

The Airy expressions for u and w are

$$u = c\,\eta\, k \frac{\cosh kz}{\sinh kD} \tag{62}$$

$$w = \frac{\partial\eta}{\partial t} \frac{\sinh kz}{\sinh kD} \tag{63}$$

Comparing with eqs. 33 and 34 it is seen that in eqs. 62 and 63 $y = D + \eta$ has been approximated by D, a correct hydrodynamic approximation in a first order theory. So the difference does not seem to be big. But for waves with a wave height of practical interest the difference can be felt important.

It is seen that the Airy expression eq. 62 will give bigger forward horizontal velocities below the crest than eq. 33. Below the trough eq. 62 will give a smaller backward velocity. This means that the Airy expressions will give a rather big resultant discharge, which is unwanted in a pure wave. Eq. 33 is without a resultant discharge as demanded in eqs 4, 8, and 9.

But eq. 33 may seem to fail in one respect. If we consider the velocity near the bottom the Airy expression, eq. 62, gives the same numerical size below the crest and the trough, while eq. 33 gives a bigger backward velocity than forward velocity. From nature it is known that the maximum forward velocity is bigger than the backward velocity, so that the sand at the bottom is moved forward until the bottom by the coast has reached the right steepness. But this does not mean that eq. 33 is not so good. Because for a real wave η of the crest is bigger than η of the trough and if that is used in eq. 33, u will be increased below the crest and decreased below the trough. Further, the sand-movement by the bottom is to a big extent caused by velocities due to friction. This is not incorporated in an expression as eq. 33 for a frictionless wave.
It is felt important that u in eq. 33 is in agreement with the principles put forward for this wave theory (e.g. no resultant water discharge). A formula not in agreement with the principles put forward is difficult to use in further hydrodynamic calculations, even though the formula incidentally in certain respects may give results that agree well with experiments or nature. It may be of this reason that previous attempts to calculate u for the shallow water cnoidal wave have not been very succesful.

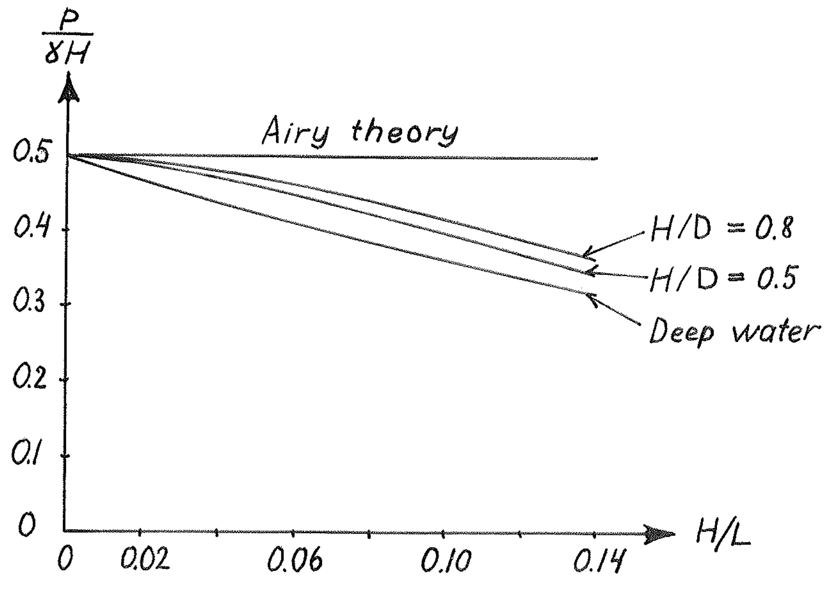

Fig. 5. The pressure at the mean water level below the crest, which is the maximum wave pressure. Comparison of the first order theory of this chapter with the Airy theory.

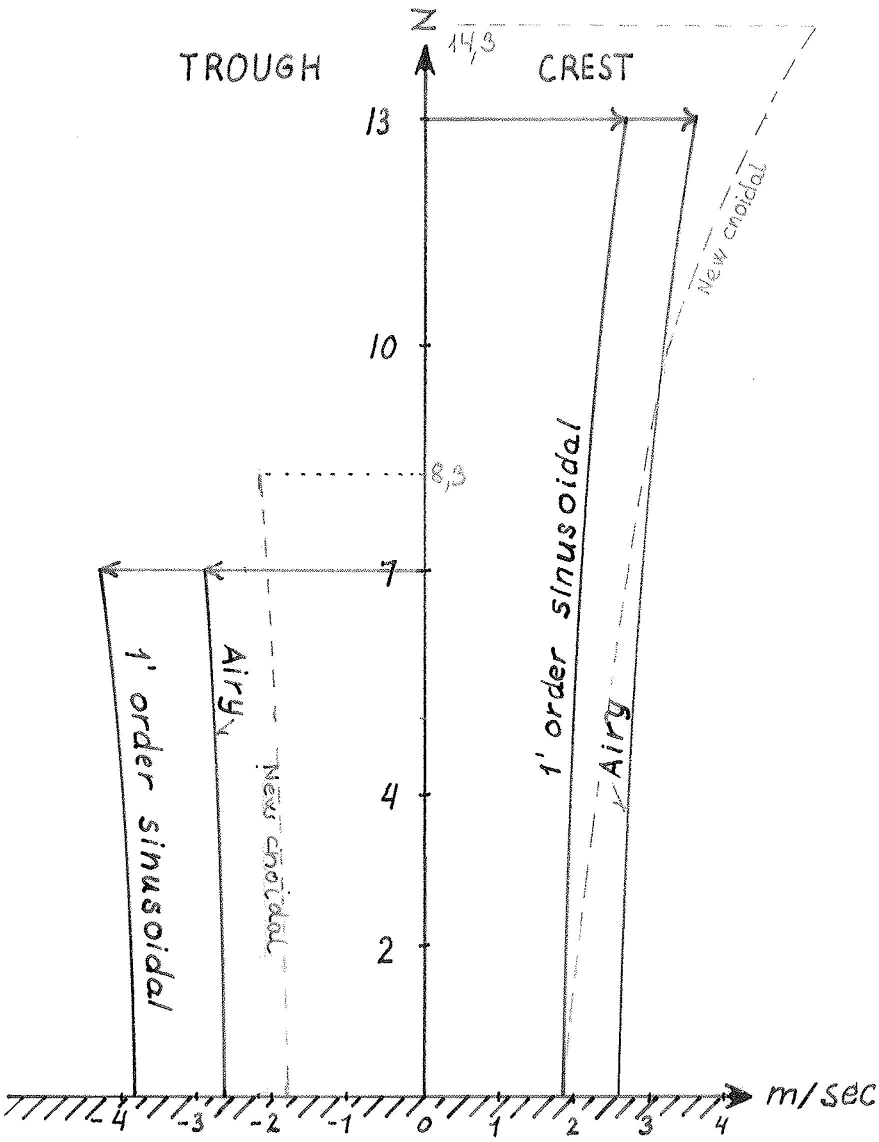

Fig. 6. The horizontal velocity, u, below the crest and the trough of the progressive wave with T = 1o sec., H = 6 metres, D = 1o metres.
Comparison of the first order theory of this chapter with the Airy theory.
It is seen that the Airy theory will give a resultant discharge, while the new first order sinusoidal expressions give no problems of this kind. But with a trough that is so deep as H/2 the new expression will give rather big backward velocities. This problem is overcome by using a more reasonable trough depth, from the cnoidal theory of chapter IX.

NUMERICAL EXAMPLE

Let us consider a wave with the period, wave height and mean water depth given as

$$T = 1o \text{ seconds} \quad ; \quad H = 6 \text{ metres} \quad ; \quad D = 1o \text{ metres} \qquad (64)$$

The wave length is then found in the usual way by the aid of wave tables for Airy waves. The deep water wave length will be

$$L_o = \frac{g}{2\pi} T^2 = 1.56 \cdot 1o^2 = 156 \text{ m} \qquad (65)$$

$$\frac{D}{L_o} = \frac{1o}{156} = o.o64 \Rightarrow \frac{D}{L} = o.1o8 \qquad (66)$$

$$L = \frac{1o}{o.1o8} = 93 \text{ m} \qquad (67)$$

The wave steepness is

$$\frac{H}{L} = \frac{6}{93} = o.o65 = 6.5 \% \qquad (68)$$

The celerity is

$$c = \frac{L}{T} = \frac{93}{1o} = 9.3 \text{ m/sec} \qquad (69)$$

Eq. 31 gives k

$$k = \frac{2\pi}{L} = \frac{2\pi}{93} = o.o67 \text{ m}^{-1} \qquad (7o)$$

The horizontal velocity at the surface, $z = y = D + \eta$, is, from eq. 33,

$$u_s = c\,\eta\, k \coth k(D + \eta) \qquad (71)$$

For the crest we get, at the surface,

$$u_{s,crest} = c \frac{H}{2} k \coth k\left(D + \frac{H}{2}\right) = 9.3 \cdot \frac{6}{2} \cdot o.o67 \cdot 1.43 = 2.7 \text{ m/sec} \qquad (72)$$

For the trough we get, at the surface

$$u_{s,trough} = - c \frac{H}{2} k \coth k\left(D - \frac{H}{2}\right) = - 4.3 \text{ m/sec} \qquad (73)$$

The horizontal velocity at the bottom, z = o, is

$$u_b = c\,\eta\,k\,\frac{1}{\sinh k(D+\eta)} \tag{74}$$

Below the crest we get, at the bottom,

$$u_{b,crest} = c\,\frac{H}{2}\,k\,\frac{1}{\sinh k(D+H/2)}$$

$$= 9.3 \cdot \frac{6}{2} \cdot 0.067 \cdot \frac{1}{0.99} = 1.9 \text{ m/sec} \tag{75}$$

Below the trough we get, at the bottom,

$$u_{b,trough} = -\,c\,\frac{H}{2}\,k\,\frac{1}{\sinh k(D-H/2)}$$

$$= -\,3.8 \text{ m/sec} \tag{76}$$

The vertical velocity is got from eq. 34. At the surface z = y we get

$$w_s = -\,c\,\frac{\partial\eta}{\partial x} \tag{77}$$

The maximum value is then found for $\eta = 0$, to

$$w_{s,max} = c\,\frac{H}{2}\,k = 9.3 \cdot \frac{6}{2} \cdot 0.067 = 1.9 \text{ m/sec} \tag{78}$$

For the pressure we use eq. 38. At the surface we get p = o. At the mean water level, z = D, eq. 38 will be

$$\frac{p}{\gamma} = \eta\left\{1 - \frac{\tanh kD}{\tanh ky}\left[1 - \frac{\cosh kD}{\cosh ky}\right]\right\} \tag{79}$$

Below the crest we get for z = D

$$\frac{p}{\gamma} = \frac{H}{2}\left\{1 - \frac{\tanh kD}{\tanh k(D+H/2)}\left[1 - \frac{\cosh kD}{\cosh k(D+H/2)}\right]\right\}$$

$$= \frac{6}{2}\left\{1 - \frac{0.58}{0.70}\left[1 - \frac{1.23}{1.40}\right]\right\} = 2.7 \text{ m} \tag{80}$$

At the bottom we find the wave pressure

$$\frac{p^+}{\gamma} = \eta\left\{1 - \frac{\tanh kD}{\tanh ky}\left[1 - \frac{1}{\cosh ky}\right]\right\} \tag{81}$$

where

$$p^+ = p - \gamma(D - z) \tag{82}$$

Below the crest we get at the bottom

$$\frac{p^+}{\gamma} = \frac{6}{2}\left\{1 - \frac{0.58}{0.70}\left[1 - \frac{1}{1.40}\right]\right\} = 2.3 \text{ m} \tag{83}$$

Below the trough we get at the bottom

$$\frac{p^+}{\gamma} = -\frac{6}{2}\left\{1 - \frac{0.58}{0.44}\left[1 - \frac{1}{1.11}\right]\right\} = -2.6 \text{ m} \tag{84}$$

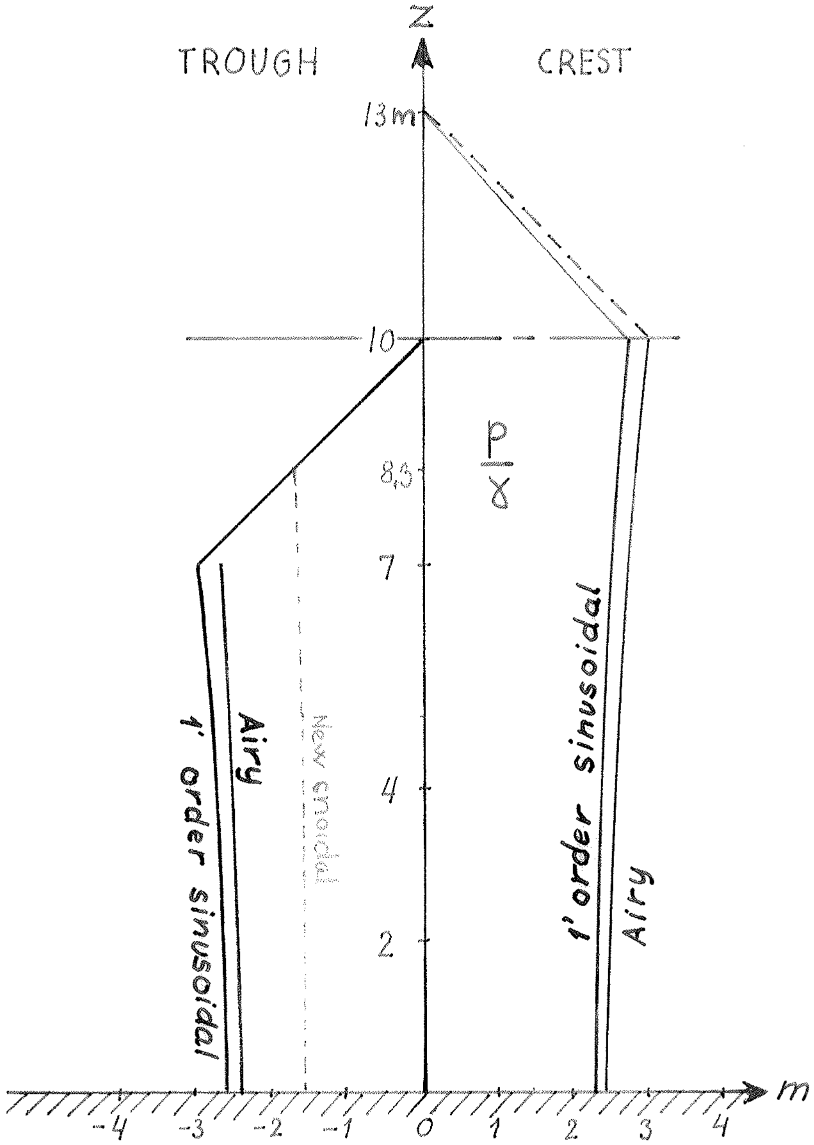

Fig. 7. The wave pressure p^+/γ below the crest and the trough of the wave with T = 1o sec, H = 6 m, D = 1o m. Comparison of the first order theory of this chapter with the Airy theory.

We will now compare this with the results of the Airy theory. We get u from eq. 62, and find

$$u_{s,crest} = c \frac{H}{2} k \frac{\cosh k(D + H/2)}{\sinh kD} = 3.6 \text{ m/sec} \tag{85}$$

to be compared to 2.7 m/sec of eq. 72

$$u_{s,trough} = -c \frac{H}{2} k \frac{\cosh k(D - H/2)}{\sinh kD} = -2.9 \text{ m/sec} \tag{86}$$

to be compared to - 4.3 m/sec of eq. 73

$$u_{b,crest} = c \frac{H}{2} k \frac{1}{\sinh kD} = 2.6 \text{ m/sec} \tag{87}$$

to be compared to 1.9 m/sec of eq. 75

$$u_{b,trough} = -u_{b,crest} = -2.6 \text{ m/sec} \tag{88}$$

to be compared to - 3.8 m/sec of eq. 76

$w_{s,max}$ will be the same as in eq. 78

The pressure is given by eq. 59. Above the mean water level $z = D$ it is not defined. Below the crest we get for $z = D$

$$\frac{p}{\gamma} = \frac{H}{2} = \frac{6}{2} = 3.0 \text{ m} \tag{89}$$

to be compared to 2.7 m of eq. 80.

At the bottom below the crest we get

$$\frac{p^+}{\gamma} = \frac{H}{2} \frac{1}{\cosh kD} = 2.4 \text{ m} \tag{90}$$

to be compared to 2.3 m of eq. 83.

Below the trough we get at the bottom

$$\frac{p^+}{\gamma} = -\frac{H}{2} \frac{1}{\cosh kD} = -2.4 \text{ m} \tag{91}$$

to be compared to - 2.6 m of eq. 84.

At the surface of the trough Airy gives

$$\frac{p^+}{\gamma} = -\frac{H}{2} \frac{\cosh k(D - H/2)}{\cosh kD} = -2.7 \text{ m} \tag{92}$$

This means that the pressure at the surface will be $p = 0.3$ m, while the theory of this chapter gives the correct $p = 0$.

PRESSURE

It is seen that the pressure as given in eq. 38 is a little complicated to use for handcalculations. It is then a question if the expression cannot be changed a little.

In view of the Airy expression of eq. 59, and comparing the expressions for u and w of eqs. 33 and 34 with eqs. 62 and 63, it is found temptating to propose

$$\frac{p}{\gamma} = D - z + \eta \frac{\cosh kz}{\cosh ky} \qquad (93)$$

which for $z \lesssim D$ and $z \lesssim y$ gives

$$\frac{p^+}{\gamma} = \frac{p}{\gamma} - (D - z) = \eta \frac{\cosh kz}{\cosh ky} \qquad (94)$$

This expression is much more simple. It is also a correct first order expression, because the difference between eqs. 38 and 94 is only of second order magnitude. Both expressions give the same wanted expression for the deep water limit.

But integrating the bottom pressure

$$\frac{p_b^+}{\gamma} = \eta \frac{1}{\cosh ky} \qquad (95)$$

over a wave length we will not get exactly zero as wanted, but a second order value. Eq. 95 gives slightly too small values for the pressure near the bottom.

Otherwise the difference between eqs. 38 and 93 is small enough to propose eq. 93 for practical use.