CHAPTER III

HISTORICAL BACKGROUND

INTRODUCTION

After we have studied the first order theory in chapter II it can be of interest to consider the classical wave hydrodynamics. The basic equations are of course the same, i.e. the equation of continuity, the equations of momentum, and the boundary conditions. But they are used in a different succession.

WAVE HYDRODYNAMICS

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = o \tag{1}$$

where $u = u(x,z,t)$ and $w = w(x,z,t)$ are the horizontal and vertical particle velocities.

The horizontal and vertical dynamic equations for a frictionless fluid are

$$-\frac{\partial p}{\partial x} = \rho \frac{du}{dt} = \rho \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right] \tag{2}$$

$$-\frac{\partial p}{\partial z} - \gamma = \rho \frac{dw}{dt} = \rho \left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \right] \tag{3}$$

$p = p(x,z,t)$ is the water pressure, ρ is the unit mass, and γ is the unit weight.

Eq. 2 is differentiated with respect to z and eq. 3 is differentiated with respect to x. $\partial^2 p/(\partial x \partial z)$ is then eliminated from the two equations, the terms are rearranged, and eq. 1 is used to give

$$\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + w\frac{\partial \Omega}{\partial z} = o \tag{4}$$

or

$$\frac{d\Omega}{dt} = o \tag{5}$$

where Ω is the rotation defined as

$$\Omega = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \tag{6}$$

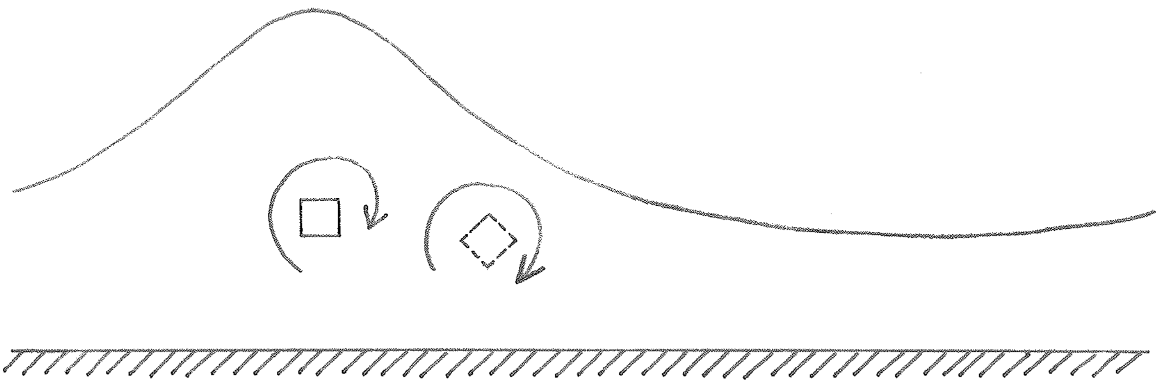

Fig. 1. The rotation of a particle must be constant in time.

Eq. 5 says that the rotation of a particle is constant in a frictionless fluid. The motion does not need to be irrotational, but for $\Omega = 0$ the classical calculations are more simple. We also see, that if the dynamic equations are fulfilled within a given order of approximations the rotational condition of eq. 5 is fulfilled within the same order of approximations.

So in hydrodynamics eq. 5 is a derived condition that does not need to be used when eqs. 2 and 3 are used in the right way. Eq. 5 can be used to check a wave theory, and then all the terms of eq. 4 must be remembered.

For a two-dimensional wave the equation of continuity, eq. 1, leads to the stream function ψ so that

$$u = -\frac{\partial \psi}{\partial z} \tag{7}$$

$$w = \frac{\partial \psi}{\partial x} \tag{8}$$

For an irrotational wave, $\Omega = 0$, eq. 6 gives

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{9}$$

the Laplace equation.

Eq. 9 is solved to give irrotationl waves. But eq. 9 can also easily be changed to include rotation by writing

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = f(\psi) \tag{10}$$

where $f(\psi)$ is a function of ψ.

This is e.g. done by Dubreil - Jacotin.

POTENTIAL THEORY

A different and much used wave theory is the potential theory. For irrotational movement, $\Omega = o$ in eq. 6, we have a velocity potential φ, so

$$u = -\frac{\partial \varphi}{\partial x} \tag{11}$$

$$w = -\frac{\partial \varphi}{\partial z} \tag{12}$$

Then the equation of continuity gives the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = o \tag{13}$$

Eqs. 2 and 3 give after some calculations the Bernoulli equation

$$-\frac{1}{g}\frac{\partial \varphi}{\partial t} + z + \frac{p}{\gamma} + \frac{1}{2g}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] = C(t) \tag{14}$$

for a fluid with $\Omega = o$. $C(t)$ is a constant dependent on the time.

At the surface we have the kinematic surface condition

$$w_s = \frac{\partial \eta}{\partial t} + u_s \cdot \frac{\partial \eta}{\partial x} \tag{15}$$

where w_s and u_s are the vertical and horizontal velocities at the surface. η is the surface elevation. This condition will though not be used exactly at the surface. In the first order theory it will be used at the mean water level, because η is assumed to be negligible. This approximation is necessary, even though it is near the surface we have the bigger changes in velocities.

The dynamic surface condition $p = o$, can in a first order theory with eq. 13 be approximated to

$$\eta - \frac{1}{g}\frac{\partial\varphi}{\partial t} = o \tag{16}$$

at the mean water level.

The kinematic and dynamic surface conditions will after a few calculations give the first order surface condition, but at the mean water level

$$\frac{\partial\varphi}{\partial z} + \frac{1}{g}\frac{\partial^2\varphi}{\partial t^2} = o \tag{17}$$

assuming φ to be periodic this can be written

$$\frac{\partial\varphi}{\partial z} - \frac{\omega^2}{g}\varphi = 0 \tag{18}$$

where ω is the angular frequency. We then end up with a mathematical problem as shown in fig. 2.

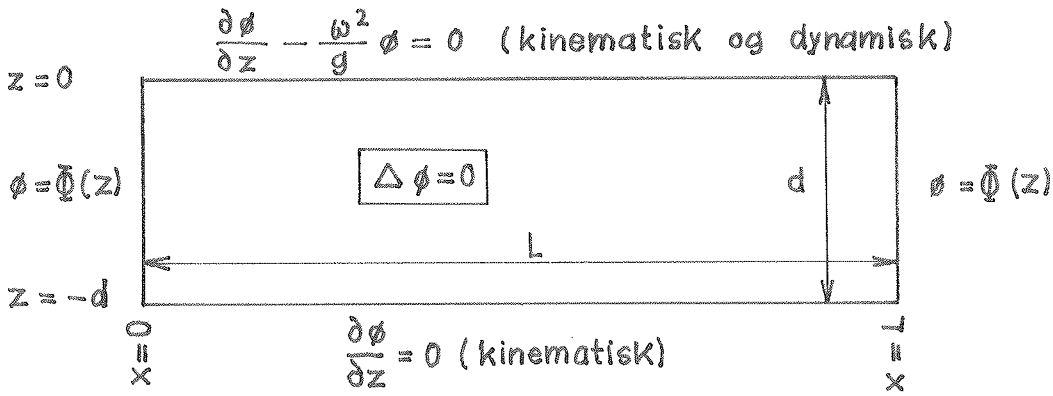

Fig. 2. In the classical theories the physical problem with water is changed to a mathematical problem, (as shown here in understandable Danish by Lundgren and Jonsson).

The solution of the potential theory gives us the classical Airy wave of first order, which is very much in use by engineers. It is apparently a great relief to introduce φ instead of u and w, whereby the number of variables is reduced. So anyone used to solve hydrodynamic problems in the mathematical way will find it natural to use φ and change the physical problem to a mathematical problem.

The method used in chapter II is rather much different. So maybe it is found complicated, and difficult to be sure of. But for persons without any specific hydrodynamic traditions the author feels that chapter II is just as easy to understand as the potential theory, and for engineers it may even be easier, because the problems are 'kept in close contact with the water'.

Within usual first order approximations the results of the two theories are in agreement, but the engineer is expected to find the new results better in some respects. Hydrodynamists may find it of interest that the number of assumptions are decreased (no demand to the rotation and the kinematic surface condition).

To improve the formulas it is possible to continue the theory to get a second order approximation, the Stokes' wave of 188o, which will give a higher and more narrow crest. Higher order waves have also been given since. They involve long calculations. The results are given in rather long formulas that are not always so handsome to use, and the numerical results may be of little value and even misleading in some practical cases.

But higher order theories are of big interest in giving a qualitative picture of the changes that should be made in the lower order theories.

With the theory of this thesis it is just as well possible to continue to higher order theories, as shown in later chapters.

SHALLOW WATER

The sinusoidal higher order theories are specially difficult to use for shallow waters. This can be a serious problem because coastal structures are often placed in shallow waters. So waves in shallow waters have had special attention for more than 1oo years and a special shallow water wave theory has been developed. In shallow water waves the wave length L is much bigger than the water depth D, so terms of higher orders in D/L can be neglected. In the canal wave theory a fundamental equation is the (integrated) equation of continuity which gives (as in chapter II)

$$\frac{\partial q}{\partial x} = - \frac{\partial \eta}{\partial t} \tag{19}$$

and

$$q = \int_{\text{bottom}}^{\text{surface}} u \, dz \tag{2o}$$

q is the water discharge through a vertical, η the surface elevation, and u the horizontal particle velocity. In e.g. the book of Stoker a page is devoted these considerations, but Stoker seems to treat the shallow water waves (the cnoidal waves) independent of those considerations at other pages.

Lundgren has also worked with the above considerations in the technical wave theory. The theory might have led to some of the waves in this thesis if working with integrated quantities was reduced where possible, and if important boundary conditions were fulfilled exactly. Although Lundgren's theory did not have any influence on the present theory, it is important to have this basis in mind, because Lundgren has been the professor for the author. Instead of asking the author to work along traditional lines using φ and ψ theories, Lundgren has accepted the more physical considerations of the wave theories here, also when they were on a more initial state.

The classical cnoidal waves, that are of second order are found by a potential theory that besides neglecting all usual higher order terms also neglects terms that are of higher orders in D/L.

The cnoidal waves were given already in 1895 by Korteweg and de Vries. But they have been difficult to use because the expressions for particle velocities and pressure were not so convincing, and because some of the elliptic functions involved may seem complicated to use without a computer.

The word 'cnoidal waves' is usually associated with shallow waters. But as shown in chapter VI we can as well have cnoidal waves on infinite deep water, and with advantage.

The wave theories considered above are all approximations of a certain 'order'. But before those theories Gerstner in 1802 gave his trochoidal wave, a deep water wave which fulfils the basic hydrodynamic conditions exactly. It has though the disadvantage that it has a rotation of second order in the direction opposite to that normally created by the wind.

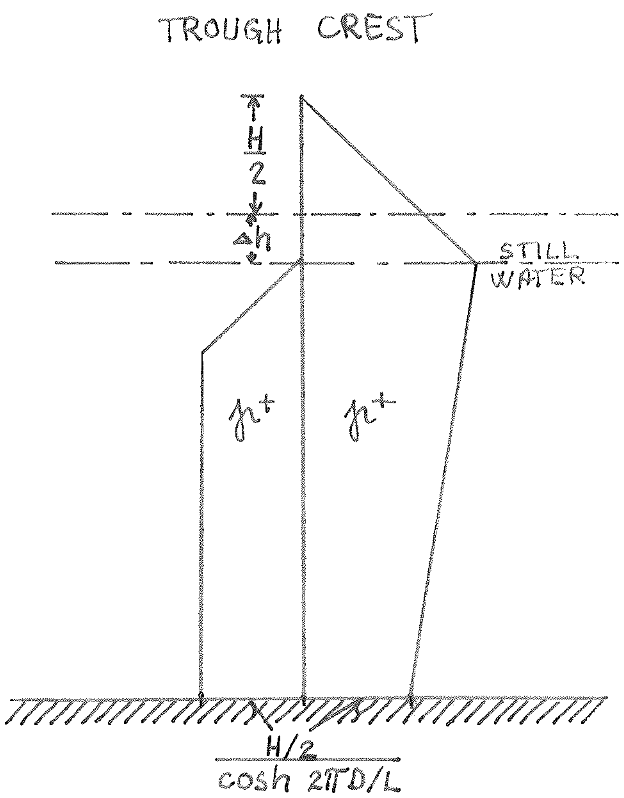

Fig. 3.
The simplified Sainflou formula.
The pressure is calculated at the bottom as shown, and the crest is calculated to be $H/2 + \Delta h$ above still water level and the trough $H/2 - \Delta h$ below still water level, where
$\Delta h = \pi H/(2L) \cosh 2\pi D/L$.
The pressure distribution is then approximated to be linear.

STANDING WAVES

The standing wave is not nearly as common in nature as the progressive wave. But it is important for the engineer. At a vertical face breakwater we often see the pure standing wave.

The first order standing wave is most simple found by superposition of two Airy waves propagating in opposite directions, the incoming wave and the reflected wave. But it can also be found directly by the potential wave theory. Like for the progressive wave the theory can be extended to higher order theories. Higher order standing sinusoidal waves reveal interesting qualities of the standing wave, but the results should not be used uncritically.

In connection with the standing waves a special interest concerns the pressure on the vertical wall. The classical expression for pressure is given by Sainflou in 1928. He worked in the lagrangian system which in some respects may be the preferred system, but for calculations of the pressure on a wall the engineer prefers the eulerian system. For practical use the simplified Sainflou formula can be used as shown in fig. 3. The Sainflou formula is reasonable, but may though in some cases yield too high pressures.

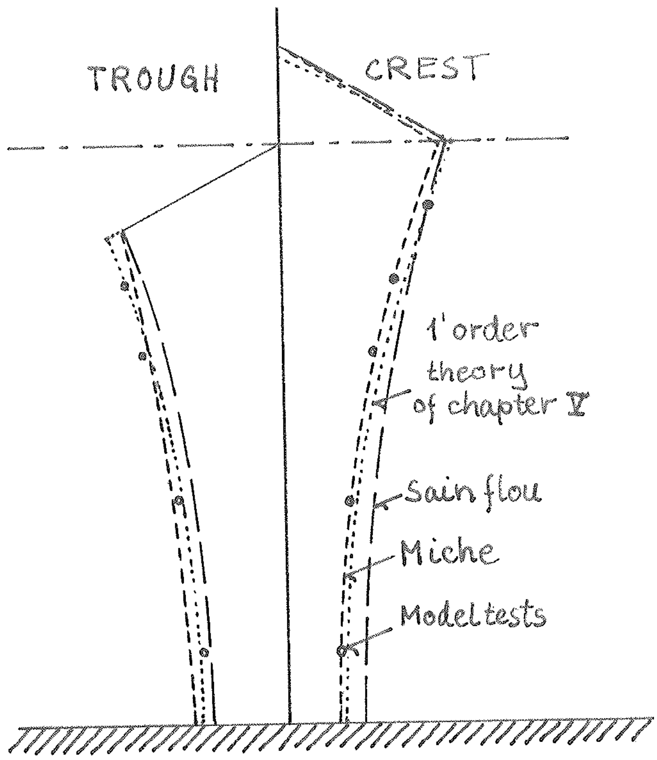

Fig. 4.
The classical comparison of the theories of Sainflou and Miche with modeltests for the wave with $H = 3$ m; $L = 80$ m; $D = 19.5$ m. Further the more simple expression from the first order theory of chapter V is shown.

In 1944 Miche gave the second order standing wave and its pressure. In some cases it gives a pressure that is better, compared to experiments, than that of Sainflou. In other cases, when the water depth (relative to the wave length) is not so big, the Miche formula is not so good.

Different theories of higher orders have since been given, e.g. in 1967 Goda gave the expressions for a fourth order sinusoidal wave. Goda is concerned with the fact that some higher order theories give a pressure at the surface that is not numerically equal to o.

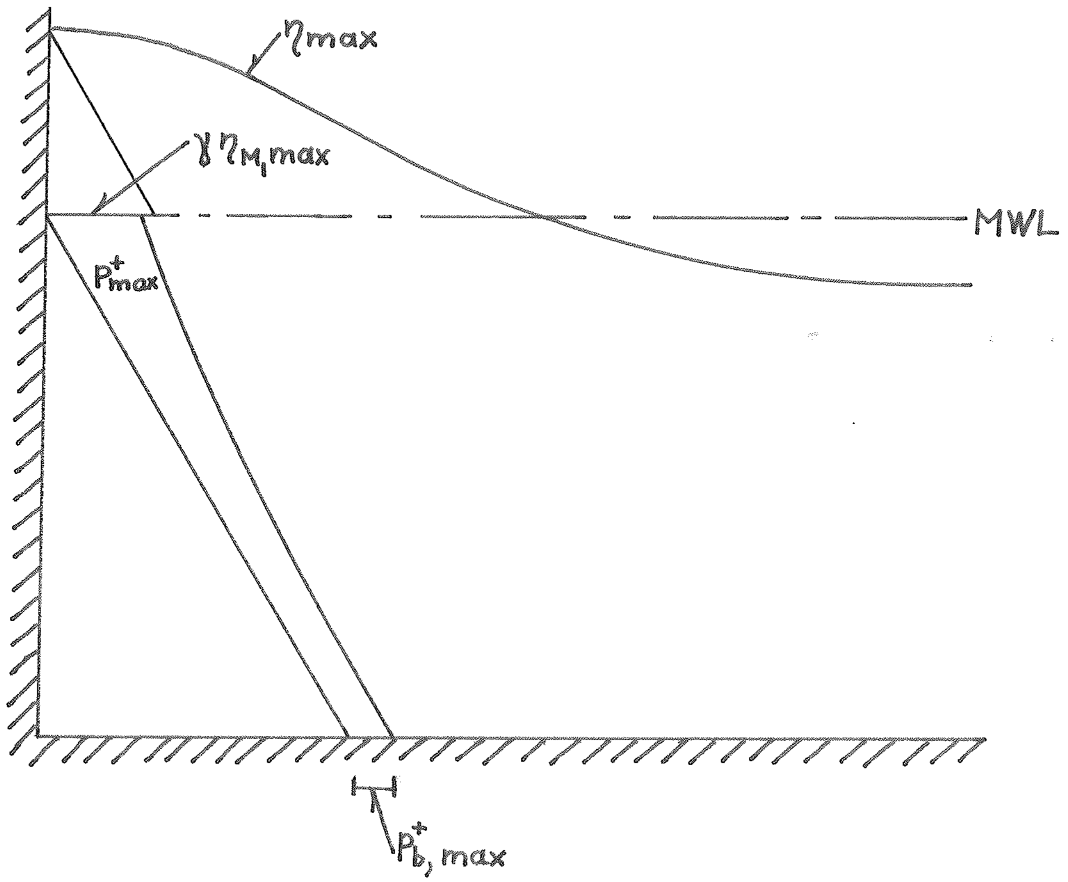

Fig.5. Pressure in a second order wave using the hydrostatic pressure above the mean water level and the Stokes' expression below. The proposal gives a discontinuity at the mean water level, which is negligible for $H/L \approx o$, but which for steeper waves (e.g. deep water waves) can give a pressure leap of 1oo %.

This problem has also been studied and partly solved in the lecture notes of the Technical University of Denmark. There the pressure below mean water level is proposed to be given by the Airy expression for the first order wave and by a Stokes' expression for the second order wave. In both cases the pressure above the mean water level is proposed to be hydrostatic with the pressure $p = o$ at the surface and $p = \eta$ at the mean water level. In fig. 6 this proposal for the Airy wave is compared to modeltests.

It is understood that when we have trough at the wall this proposal will not give the pressure $p = o$ at the surface. To propose a hydrostatic pressure above the mean water level means that the vertical acceleration is neglected where it is biggest. This is hydrodynamically correct in a first order theory but will give too high pressures, e.g. at the mean water level, specially for the second order wave.

The vertical acceleration of the water can be so important that we do not have the maximum pressure at the time of crest even at rather small depths. This problem has been considered closer by e.g. Zagrjadskaja. It is also a result of the theory of second order of e.g. Miche.

There has been written a lot about waves all over the world because they are so essential in coastal hydraulics and in ocean engineering. Only a few papers have been referred above because they are found of interest in connection with the work of this thesis. There are a number of books that treats the subject, some in a mathematical way (e.g. ref.[8]) and some including the more practical aspects (e.g. ref.[23]). The latest proceedings from symposiums on coastal hydraulics usually have some papers on waves, with a number of references (e.g. the proceedings refs. [13] and [22] belong to).

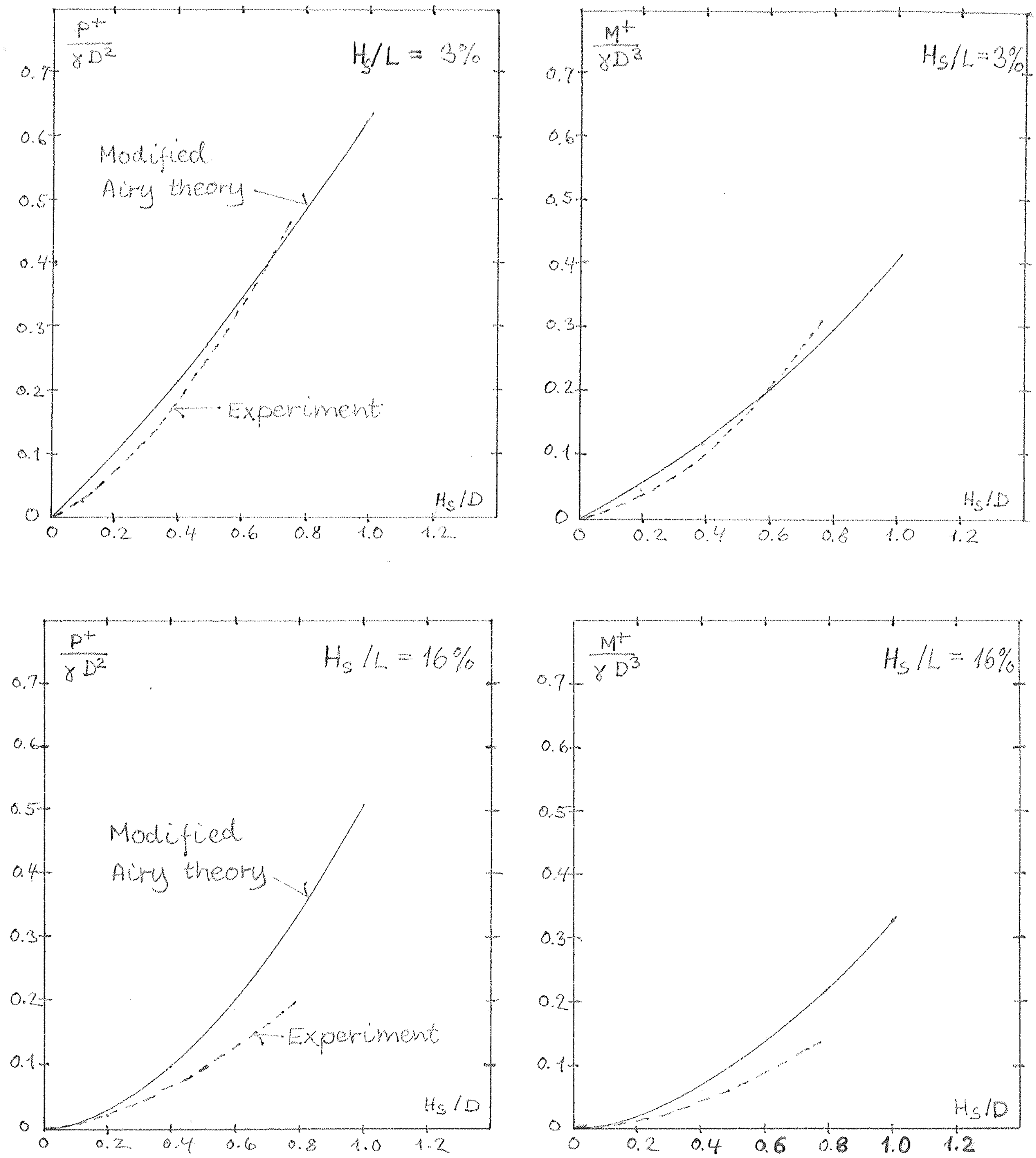

Fig. 6. The total horizontal force P^+ and the overturning moment M^+ from the wave pressure p^+ on a vertical wall from standing waves. The Airy theory, with hydrostatic pressure above the mean water level, compared to model tests. Compare with figs. 6 and 7 of chapter V.

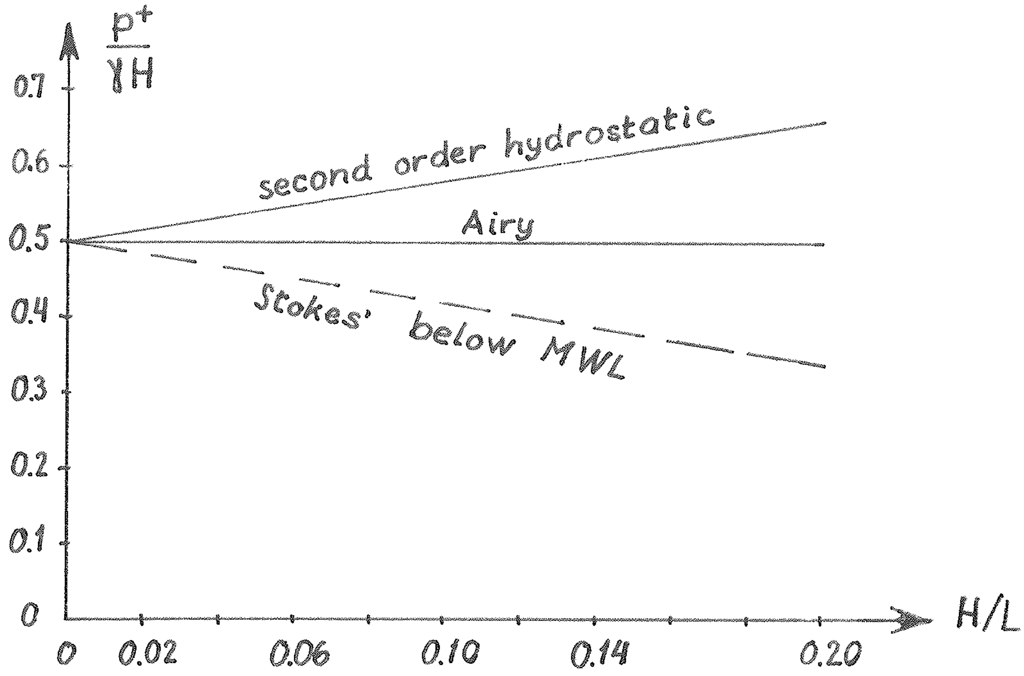

Fig. 7. Second order pressure at the mean water level in progressive and standing deep water waves. If, as proposed in the literature, the hydrostatic pressure is used above the mean water level in a second order theory the pressure will be bigger than according to the Airy theory. This is of course a correct first order expression, but not so suitable.