CHAPTER II

PROGRESSIVE FIRST ORDER DEEP WATER WAVE

ABSTRACTS

In this chapter we will find the first order sinusoidal wave on infinite deep water in a different way than usually. The wave we will find is the same as the wellknown and much used Airy wave. But the expressions for velocities and pressure etc. will in a natural way get a deviation of second order, by which they will fulfil the boundary conditions as discussed in the preceding chapter. This gives a numerical difference of importance for the engineer, but hydrodynamically the two sets of results are identical within the first order approximation. The deep water wave is of special interest because it gives more simple expressions than the wave on arbitrary depth, and still it shows the basic procedure.

BASIC EQUATIONS

We consider two dimensional progressive waves of permanent form on an incompressible and frictionless fluid without surface tension.

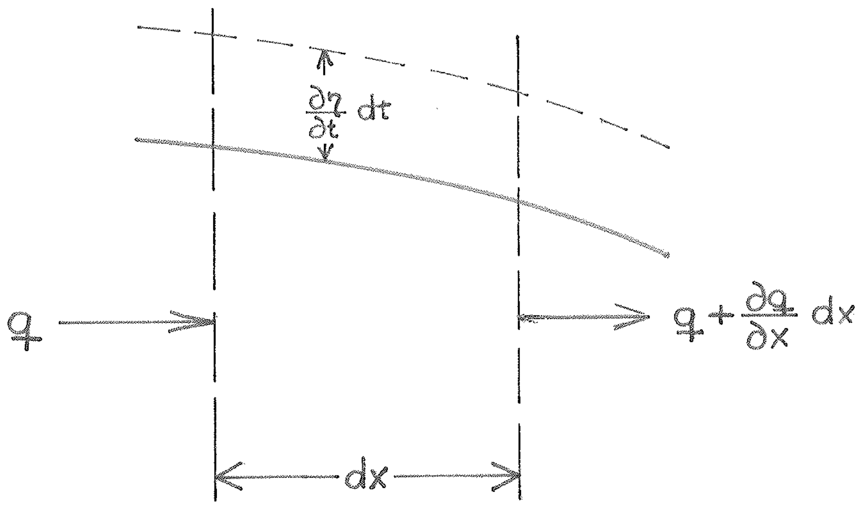


Fig. 1. The equation of continuity for the water discharge, q.

Through a stationary vertical we have the water discharge $q = q(x,z,t)$. At the same vertical the water surface moves up and down with the elevation $\eta = \eta(x,t)$. Considering fig. 1 the equation of continuity for q gives

$$\frac{\partial q}{\partial x} = -\frac{\partial \eta}{\partial t} \tag{1}$$

This expression is wellknown from the canal wave theory.

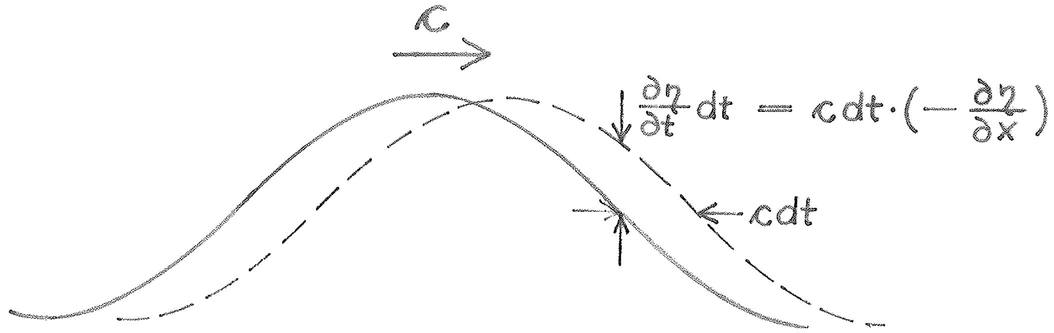


Fig. 2. A progressive wave of constant form.

From fig. 2 we get the wellknown expression for progressive waves of permanent form

$$-\frac{\partial \eta}{\partial t} = c\frac{\partial \eta}{\partial x} \tag{2}$$

where c is the celerity.

Eqs. 1 and 2 are combined and integrated to give, for a wave without a resultant discharge

$$q = c\,\eta \tag{3}$$

η has $\eta = 0$ at the mean water level, so

$$\int_0^L \eta \, dx = 0 \tag{4}$$

so that by eq. 2 we also have

$$\int_0^T \eta \, dt = 0 \tag{5}$$

L is the wave length and T the wave period, so

$$L = c\,T \tag{6}$$

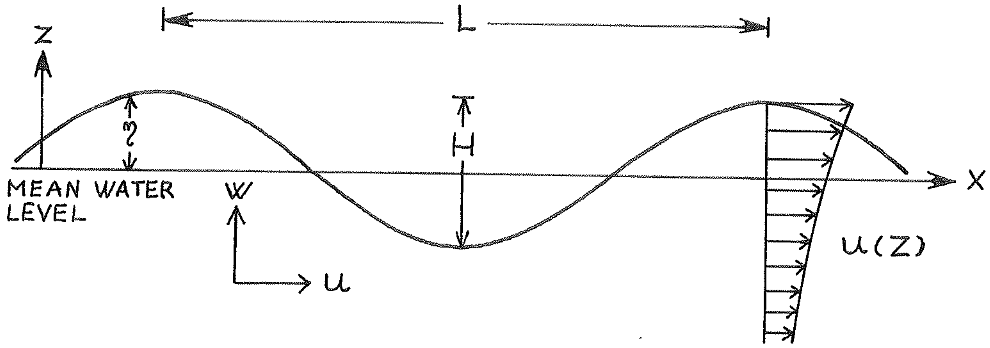


Fig. 3 Definition sketch

As mentioned in chapter I, eq. 3 is a rather wellknown expression for the engineer. With a given vertical distribution of the horizontal particle velocity $u = u(x,z,t)$, it is possible to find q also by integration of u from the bottom up to the surface

$$q = \int_{-\infty}^{\eta} u \, dz \tag{7}$$

So apart from the vertical distribution u is determined (as a function of η). By physical considerations it is possible to narrow the number of distribution functions . But in this chapter we only need to say that from our knowledge of the classical Airy theory we can expect that u is exponential distributed. So to fulfil eq. 7

we write for u

$$u = q\ R\ e^{R(z-\eta)} \qquad (8)$$

R is an unknown constant, which later (eq. 35) is found to $R = 2\pi/L$, like for the Airy wave.

In chapter VI u will be chosen more arbitrary, but in this first development of the theory we want to make it more simple and only prove that we can get a first order wave with u given as in eq. 8. With eq. 3 we change eq. 8 to

$$u = c\ \eta\ R\ e^{R(z-\eta)} \qquad (9)$$

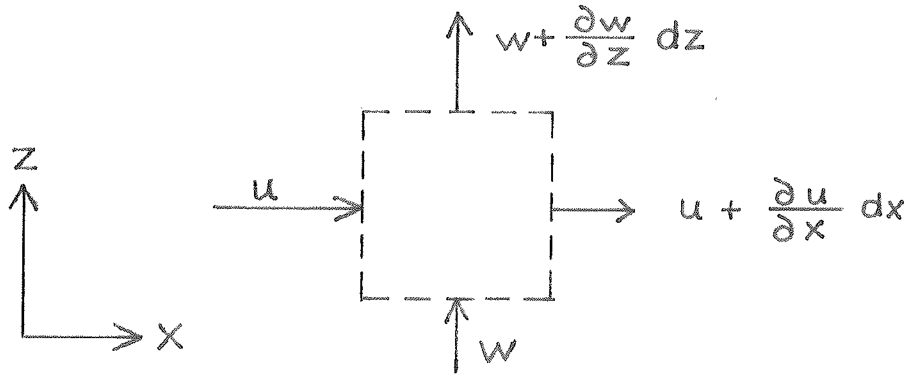


Fig. 4. The equation of continuity at a point of the fluid. An infinitesimal unit cube (dx = 1, dz = 1) is considered.

At any point of the fluid we have the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (10)$$

where $w = w(x,z,t)$ is the vertical particle velocity.

From eq. 9 we get

$$\frac{\partial u}{\partial x} = c\frac{\partial \eta}{\partial x} R\, e^{R(z-\eta)} - c\eta R^2 e^{R(z-\eta)} \frac{\partial \eta}{\partial x} \tag{11}$$

Deep below the surface the velocities due to the wave will vanish. So with the boundary condition $w \to o$ for $z \to -\infty$ eqs. 1o and 11 give by integration

$$w = c \cdot \frac{\partial \eta}{\partial x} [-1 + \eta R] e^{R(z-\eta)} \tag{12}$$

In this expression we have terms of different orders of magnitude. The following calculations will be more simple if we evaluate the terms right away and neglect minor terms. (In chapter VI all the terms are taken along as long as possible.)

Later, in eqs. 34 and 35, we will find the well known Airy solution for η

$$\eta = \frac{H}{2} \cos k(x - ct) \tag{13}$$

and for R, as mentioned

$$R = k = \frac{2\pi}{L} \tag{14}$$

H is the wave height.

So the maximum value of ηR will be

$$\eta R \leq \pi \frac{H}{L} \tag{15}$$

From measurements in the nature and the laboratory it is known that the steepness H/L will not exceed o.14. So for the maximum wave we have

$$\eta R < o.4 \tag{16}$$

In eq. 12 ηR is compared to 1. It is seen that ηR is less than 1, but to feel it justified to neglect ηR it can be necessary in eq. 15 to demand H/L even smaller. Multiplying through with $\partial \eta / \partial x$ in eq. 12 we get the two terms

$$\frac{\partial \eta}{\partial x} \quad \text{and} \quad \frac{\partial \eta}{\partial x} \eta R$$

From eq. 13 it is seen that $\partial\eta/\partial x$ is proportional to $(H/L)^1$ and $(\partial\eta/\partial x)\eta R$ is proportional to $(H/L)^2$. So $\partial\eta/\partial x$ is said to be small of 1'order and $(\partial\eta/\partial x)\eta R$ is small of 2'order. In a first order theory we will only take along the terms of first order magnitude and neglect the terms of second order (and higher order). So in eq. 12 we will write w as

$$w = -c\,\frac{\partial\eta}{\partial x}\,e^{R(z-\eta)} \qquad (17)$$

Eq. 17 will be used in the following calculations. For practical use we can choose between eqs. 12 and 17 for w . Generally eq. 12 cannot be said to be more exact than eq. 17 as long as η only is determined with a first order approximation as in eq. 13. But in certain cases eq. 12 is more satisfactory for the engineer, because it not only fulfils the equation of continuity exactly, but also the kinematic surface condition (see fig. 5).

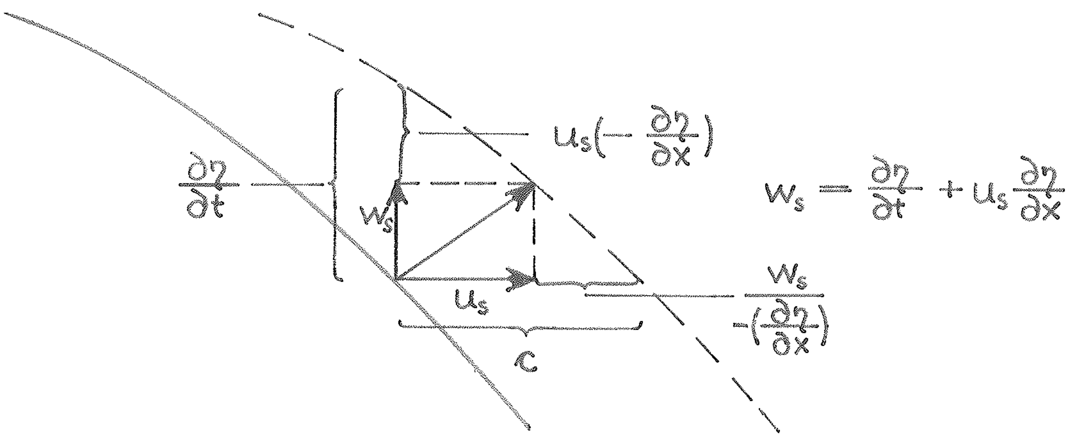


Fig. 5. u and w from eqs. 9 and 12 fulfil the kinematic surface condition exactly.

For a particle in the surface, which must stay in the surface, we have from eqs. 12, 2, and 9 for $z = \eta$

$$w_s = -c\frac{\partial\eta}{\partial x} + c\frac{\partial\eta}{\partial x}\eta R = \frac{\partial\eta}{\partial t} + u_s\frac{\partial\eta}{\partial x} \tag{18}$$

where w_s and u_s are w and u at the surface.

After having discussed the approximations that led to the first order expression, eq. 17, we will in the same way find the horizontal particle acceleration $G_x = G_x(x,z,t)$ to the first order. With eq. 2, eqs. 9 and 17 give

$$G_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -c^2\frac{\partial\eta}{\partial x}R\, e^{R(z-\eta)} \tag{19}$$

In chapter VI the exact expression for G_x is given. It is seen that in eq. 19 we have only neglected the second order term

$$c^2\frac{\partial\eta}{\partial x}\eta R^2 e^{R(z-\eta)}$$

For the engineer who would like to use the simple expressions for waves that are higher than infinitely small it can be of interest to know how many negligible terms have been neglected or how much they can add up to in numerical cases. In eq. 19 the first order approximation will be of same numerical size as in eq. 17.

The vertical particle acceleration $G_z = G_z(x,z,t)$ will in the first order be

$$G_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = c^2\frac{\partial^2\eta}{\partial x^2}e^{R(z-\eta)} \tag{20}$$

Here some terms of second and third order have been neglected, so it is more difficult in a glance to determine the consequenses in practical situations. In chapter V we will find the equivalent expression for the standing wave, and then it is seen that the first order expression is exact in the most important case, i.e. the acceleration at the surface at the vertical wall.

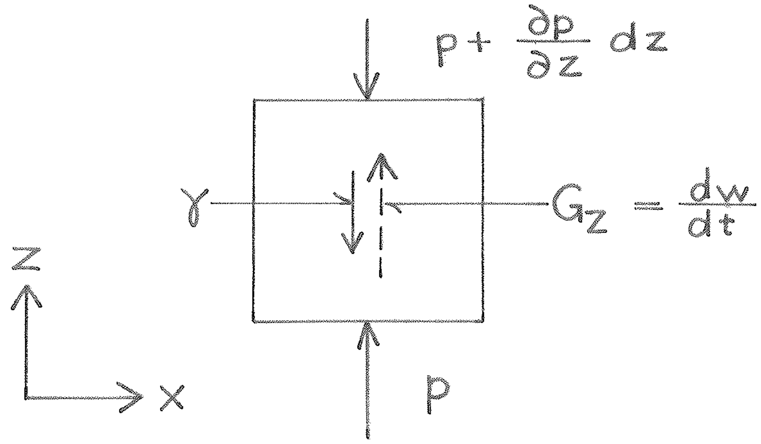


Fig. 6. The vertical equation of momentum for an infinitesimal unit cube (dx = 1, dz = 1).

The vertical dynamic equation for a frictionless fluid is

$$-\frac{\partial p}{\partial z} - \gamma = \varrho G_z \tag{21}$$

where $p = p(x,z,t)$ is the pressure of the fluid (above atmospheric pressure), g the acceleration of gravity, γ is the unit weight and ϱ the unit mass of the water, so $\gamma = \varrho g$.

Eq. 21 is integrated, and using $p = o$ at the surface $z = \eta$, we get the first order expression

$$\frac{p}{\gamma} = \eta - z + \frac{c^2}{g}\frac{\partial^2\eta}{\partial x^2}\frac{1}{R}\left[1 - e^{R(z-\eta)}\right] \tag{22}$$

In eq. 2o the acceleration, G_z, at the surface, $z = \eta$, is called G_{zs}. Then eq. 22 can also be written

$$\frac{p}{\gamma} = \eta - z + \frac{G_{zs}}{g}\frac{1}{R}\left[1 - e^{R(z-\eta)}\right] \tag{23}$$

In this way we get an expression for the pressure which may be used more general than just for regular first order sinusoidal waves, if only G_{zs} and R can be estimated reasonable.

By differentation of eq. 22 we find a first order expression for the horizontal pressure gradient

$$\frac{1}{\gamma}\frac{\partial p}{\partial x} = \frac{\partial \eta}{\partial x} + \frac{c^2}{g}\frac{\partial^3 \eta}{\partial x^3}\frac{1}{R}\left[1 - e^{R(z-\eta)}\right] \tag{24}$$

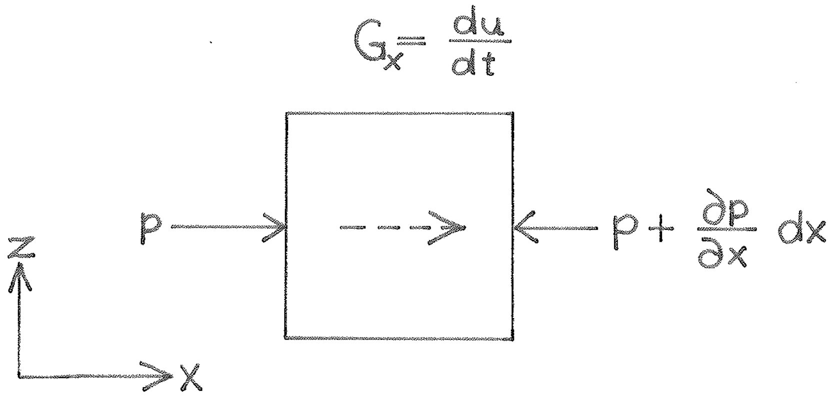


Fig. 7. The horizontal equation of momentum for an infinitesimal unit cube (dx = 1, dz = 1).

The horizontal dynamic equation is

$$-\frac{\partial p}{\partial x} = \rho G_x \tag{25}$$

which with eq. 19 gives us still an expression for $\partial p/\partial x$

$$\frac{1}{\gamma}\frac{\partial p}{\partial x} = \frac{c^2}{g}\frac{\partial \eta}{\partial x} R\, e^{R(z-\eta)} \tag{26}$$

$\partial p/\partial x$ is eliminated from eqs. 24 and 26 and we get a governing wave equation

$$\frac{\partial \eta}{\partial x} + \frac{c^2}{g}\frac{\partial^3 \eta}{\partial x^3}\frac{1}{R}\left[1 - e^{R(z-\eta)}\right] - \frac{c^2}{g}\frac{\partial \eta}{\partial x} R e^{R(z-\eta)} = 0 \tag{27}$$

From this equation we will find η, c and R.

WAVE SOLUTION

Eq. 27 is written

$$\left\{\frac{g}{c^2}\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\right\}-\left\{\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}e^{R(z-\eta)}+\frac{\partial\eta}{\partial x}Re^{R(z-\eta)}\right\}=0 \qquad (28)$$

By this the terms are assembled in two groups : one group that does not depend on z, and one group that depends on z. Each of the two groups will be set equal to o so

$$\left\{\frac{g}{c^2}\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\right\}=0 \qquad (29)$$

and

$$\left\{\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}e^{R(z-\eta)}+\frac{\partial\eta}{\partial x}Re^{R(z-\eta)}\right\}=0 \qquad (3o)$$

So eq. 28 is substituted by eqs. 29 and 3o.

It is evident that when both eq. 29 and eq. 3o are fulfilled simultaneously then eq. 28 is also fulfilled. This method of splitting an equation into two or more equations is not unknown from other problems in hydrodynamics. In eq. 3o $e^{R(z-\eta)}$ could right away be divided over, by which z would disappear, but still the equation originates from the z-dependent terms, and because of that it will here be called the z-dependent equation.
From eq. 3o we get

$$\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}=-\frac{\partial\eta}{\partial x}R \qquad (31)$$

This is substituted into eq. 29

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x}-\frac{\partial\eta}{\partial x}R=0 \qquad (32)$$

Instead of eqs. 29 and 3o we can then solve eqs. 3o and 32. By comparison it is seen that we can get eq. 32 directly from eq. 27 by substituting $z = \eta$ into eq. 27, i.e. by using the wave equation at the surface. This does of course not mean that the resulting wave will fulfil the hydrodynamic conditions only at the surface. They are fulfilled at any point within first order approximation.

A progressive wave of permanent form is given by

$$\eta = f(x - c\ t) \tag{33}$$

where f is a function. From eq. 30 it is seen that this function is a harmonic function, so for η we choose

$$\eta = \frac{H}{2} \cos R(x - ct) \tag{34}$$

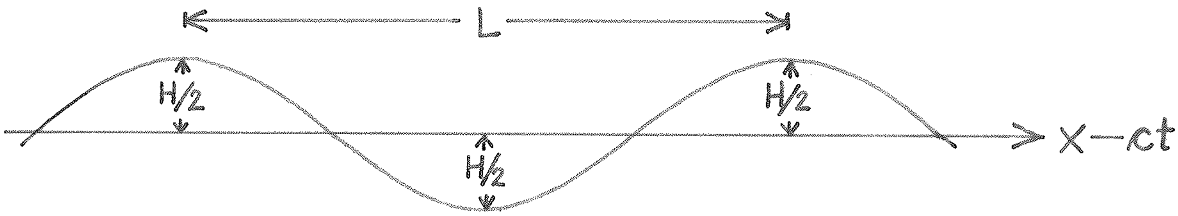


Fig.8. The solution to η is a harmonic (a cosine) function. Then the amplitude must be half the wave height and the wave length is as shown. This gives the solution for R, the original unknown constant in the vertical distribution of u.

Then R in the cos-function is determined geometrically to

$$R = \frac{2\pi}{L} = k \tag{35}$$

We have here used the usual notation k for $2\pi/L$.

Eqs. 29 and 34 then gives us the celerity

$$c = \sqrt{\frac{g}{R}} = \sqrt{\frac{g}{k}} = \sqrt{\frac{gL}{2\pi}} \tag{36}$$

This expression could earlier have been found from eq. 32, the wave equation at the surface.

So far we have said nothing about the rotation. The rotation plays a decisive role in the classical wave theory, because it is a fundamental condition to have irrotational motion. So we will here find the first order rotation from eqs. 9 and 17 to

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \tag{37}$$

If a positive or negative rotation is wanted, this can be obtained by a change in u in eq. 9. This subject is looked upon in chapter VI for deep water waves of second order.

We have now solved our wave problem. u is given already in eq. 9. At that time we did not know c, η, and R. But they have since been given by eqs. 36, 34, and 35. The same can be said about w and p in eqs. 12 and 22.

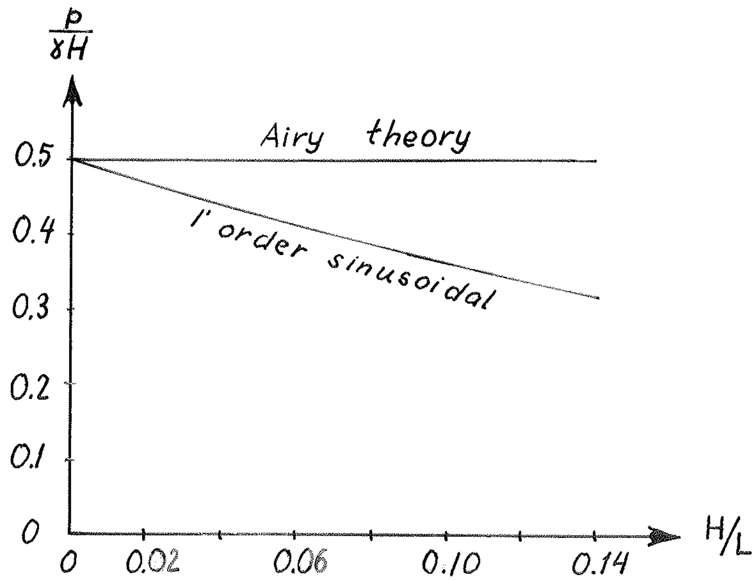


Fig. 9. Pressure at the mean water-level, z = o, below the crest. The first order sinusoidal theory of this chapter is compared with the first order Airy theory. The Airy theory neglects the vertical acceleration of the water above the mean water level, so the pressure at this point, with the maximum wave pressure, is hydrostatic. The shown first order pressure will not be changed so much by the cnoidal theory of chapter VI, because with an increased crest the pressure reducing vertical acceleration will also be increased.

FIRST ORDER DEEP WATER WAVE FORMULAS

We will here give a review of the most important expressions to be used in practice.

$$\eta = \frac{H}{2}\cos k(x-ct) \tag{38}$$

$$k = \frac{2\pi}{L} \tag{39}$$

$$c = \sqrt{\frac{gL}{2\pi}} = \sqrt{\frac{g}{k}} \tag{40}$$

$$L = cT \quad \text{so} \quad L = \frac{g}{2\pi}T^2 \tag{41}$$

$$u = c\eta k e^{k(z-\eta)} \tag{42}$$

$$w = -c\frac{\partial\eta}{\partial x}e^{k(z-\eta)} = \frac{\partial\eta}{\partial t}e^{k(z-\eta)} \tag{43}$$

$$\frac{p}{\gamma} = \eta - z + \frac{c^2}{g}\frac{\partial^2\eta}{\partial x^2}\frac{1}{k}\left[1-e^{k(z-\eta)}\right] \tag{44}$$

$$= -z + \eta e^{k(z-\eta)} \tag{45}$$

$$\frac{\partial\eta}{\partial x} = -\frac{H}{2}k\sin k(x-ct) = -\frac{1}{c}\frac{\partial\eta}{\partial t} \tag{46}$$

$$\frac{du}{dt} = ckw \tag{47}$$

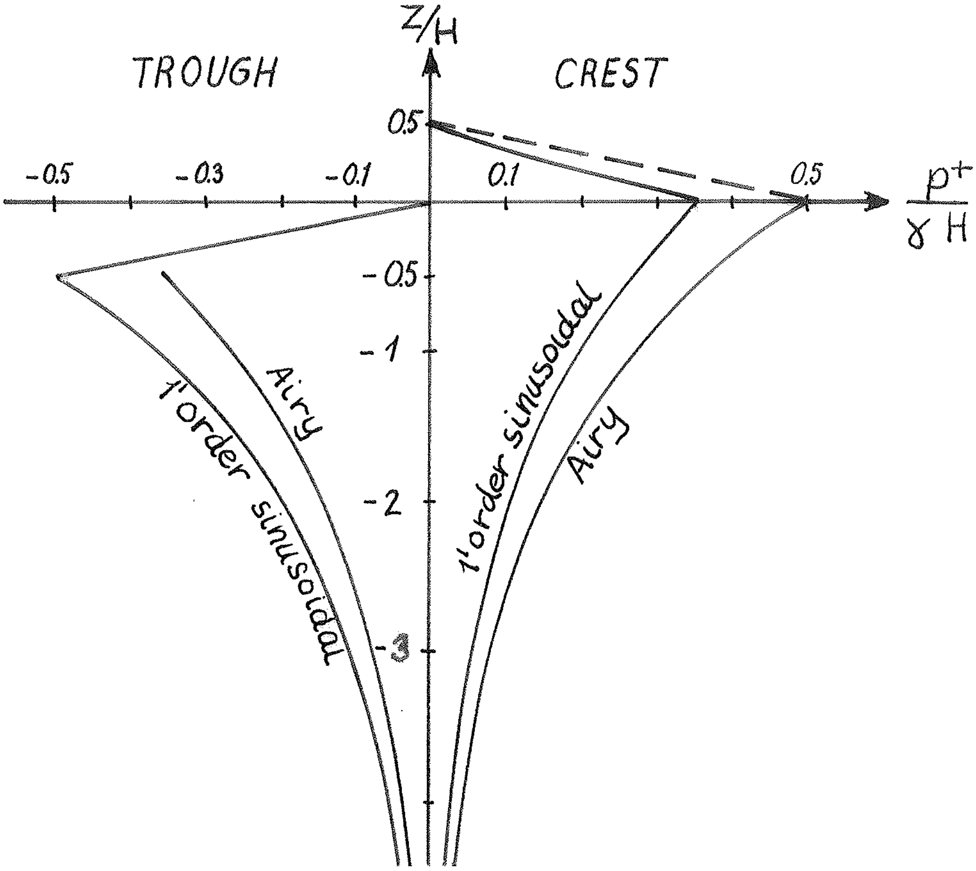


Fig. 1o. Maximum and minimum wave pressure, i.e. for crest and trough. H/L_o = 1o%. Comparison of the first order theory of this chapter with the Airy theory. The Airy expression can actually not be used above z = o for the crest, but with a hydrostatic pressure at z = o it is reasonable to continue above z = o with the hydrostatic pressure as shown by the dotted line.
At the surface of the trough the Airy expression does not give a fluid pressure of p = o (i.e. $p^+ = o.5\gamma H$), so it is more difficult to decide on a proposal for the wave pressure from the surface of the trough up to z = o.

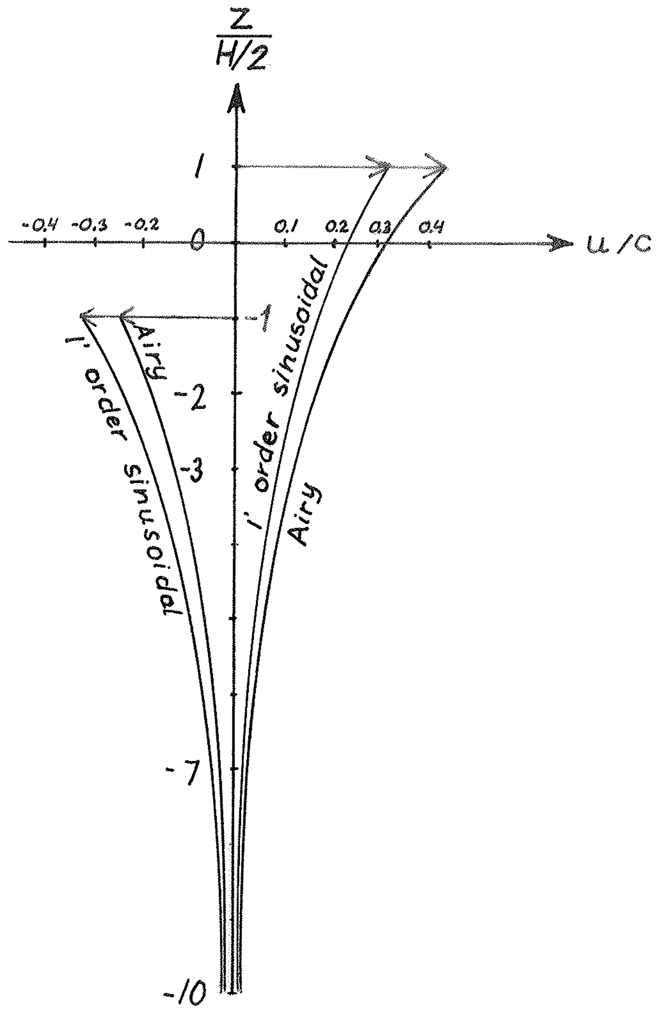


Fig. 11. Horizontal velocity below crest and trough. H/L = 10%. Comparison of the first order theory of this chapter with the Airy theory. It is seen that the Airy theory will result in a net flow.

APPENDIX II

NUMERICAL EXAMPLE

Let us illustrate the use of the final formulas, eqs. 38 to 47 with a numerical example. Let the wave period and wave height be given as

$$T = 10 \text{ seconds} \qquad H = 10 \text{ metres} \tag{48}$$

The wave length is found by eq. 41

$$L = \frac{g}{2\pi} T^2 = 1.56 \cdot 10^2 = 156 \text{ m} \tag{49}$$

The celerity is then

$$c = \frac{L}{T} = \frac{156}{10} = 15.6 \text{ m/sec} \tag{50}$$

The wave steepness

$$\frac{H}{L} = \frac{10}{156} = 0.064 = 6.4 \% \tag{51}$$

For k we find, eq. 39,

$$k = \frac{2\pi}{L} = \frac{2\pi}{156} = 0.0403 \text{ m}^{-1} \tag{52}$$

The horizontal velocity is given by eq. 42. At the surface $z-\eta = 0$ so

$$u_s = c \, \eta \, k \tag{53}$$

At the surface of the crest we get

$$u_{s,crest} = c \frac{H}{2} k = 15.6 \cdot \frac{10}{2} \cdot 0.0403 = 3.14 \text{ m/sec} \tag{54}$$

And at the surface of the trough we get

$$u_{s,trough} = - c \frac{H}{2} k = - 3.14 \text{ m/sec} \tag{55}$$

10 m below the mean water level, $z = -10$ m, we find below the crest

$$u_{crest,z=-10} = c \frac{H}{2} k \, e^{k(-10-H/2)}$$

$$= 15.6 \cdot \frac{10}{2} \cdot 0.0403 \cdot e^{-0.0403 \cdot 15}$$

$$= 3.14 \cdot 0.55 = 1.72 \text{ m/sec} \tag{56}$$

and below the trough

$$u_{trough,z=-1o} = -c\,\frac{H}{2}\,k\,e^{k(-1o+H/2)}$$

$$= -3.14 \cdot o.82 = -2.57 \text{ m/sec} \tag{57}$$

The vertical velocity is found from eq. 43. At the surface we get

$$w_s = -c\,\frac{\partial\eta}{\partial x} = \frac{\partial\eta}{\partial t} \tag{58}$$

For η = o we get the maximum value of w_s

$$w_{s,\eta=o} = c\,\frac{H}{2}\,k\cdot 1 = 3.14 \text{ m/sec} \tag{59}$$

The pressure is found from eq. 45. At the surface, $z = \eta$, we get $p=o$. At the mean water level, z = o, we get

$$\frac{p}{\gamma} = \eta\,e^{-k\eta} \qquad \text{for } z = o \tag{6o}$$

Below the crest we find

$$\frac{p}{\gamma} = \frac{H}{2}\,e^{-kH/2} = 5.o \cdot e^{-o.o4o3\cdot 5.o} = 5.o \cdot o.82 = 4.1 \text{ m} \tag{61}$$

Below the mean water level the wave pressure is

$$p^+ = p + \gamma\,z \tag{62}$$

At the surface of the trough, z = - H/2, where p = o we then find

$$\frac{p^+}{\gamma} = -\frac{H}{2} = -5.o \text{ m} \tag{63}$$

We can now calculate the same example with the Airy theory. We then have for u, w, and p^+

$$u = c\,\eta\,k\,e^{kz} \tag{64}$$

$$w = -c\,\frac{\partial\eta}{\partial x}\,e^{kz} \tag{65}$$

$$\frac{p^+}{\gamma} = \eta\,e^{kz} \tag{66}$$

So the difference between the Airy expressions and the expressions of this chapter is mainly that the Airy theory uses the depth below the mean water level, while it here is found better to use the depth below the surface.

The Airy expressions give

$$u_{s,crest} = 3.14 \cdot 1.22 = 3.84 \text{ m/sec} \tag{67}$$

to be compared with 3.14 m/sec of eq. 54.

$$u_{s,trough} = - 3.14 \cdot 0.82 = - 2.56 \text{ m/sec} \tag{68}$$

to be compared with - 3.14 m/sec of eq. 55.

$$u_{crest,z=-10} = 3.14 \cdot 0.67 = 2.10 \text{ m/sec} \tag{69}$$

$$u_{trough,z=-10} = - 2.10 \text{ m/sec} \tag{70}$$

to be compared with eqs. 56 and 57.

$w_{s,\eta=0}$ will be the same as found in eq. 59.

The pressure at the surface of the crest cannot be found by the Airy expression, eq. 66. At the mean water level we get

$$\frac{p^+}{\gamma} = \frac{p}{\gamma} = 5.0 \text{ m} \tag{71}$$

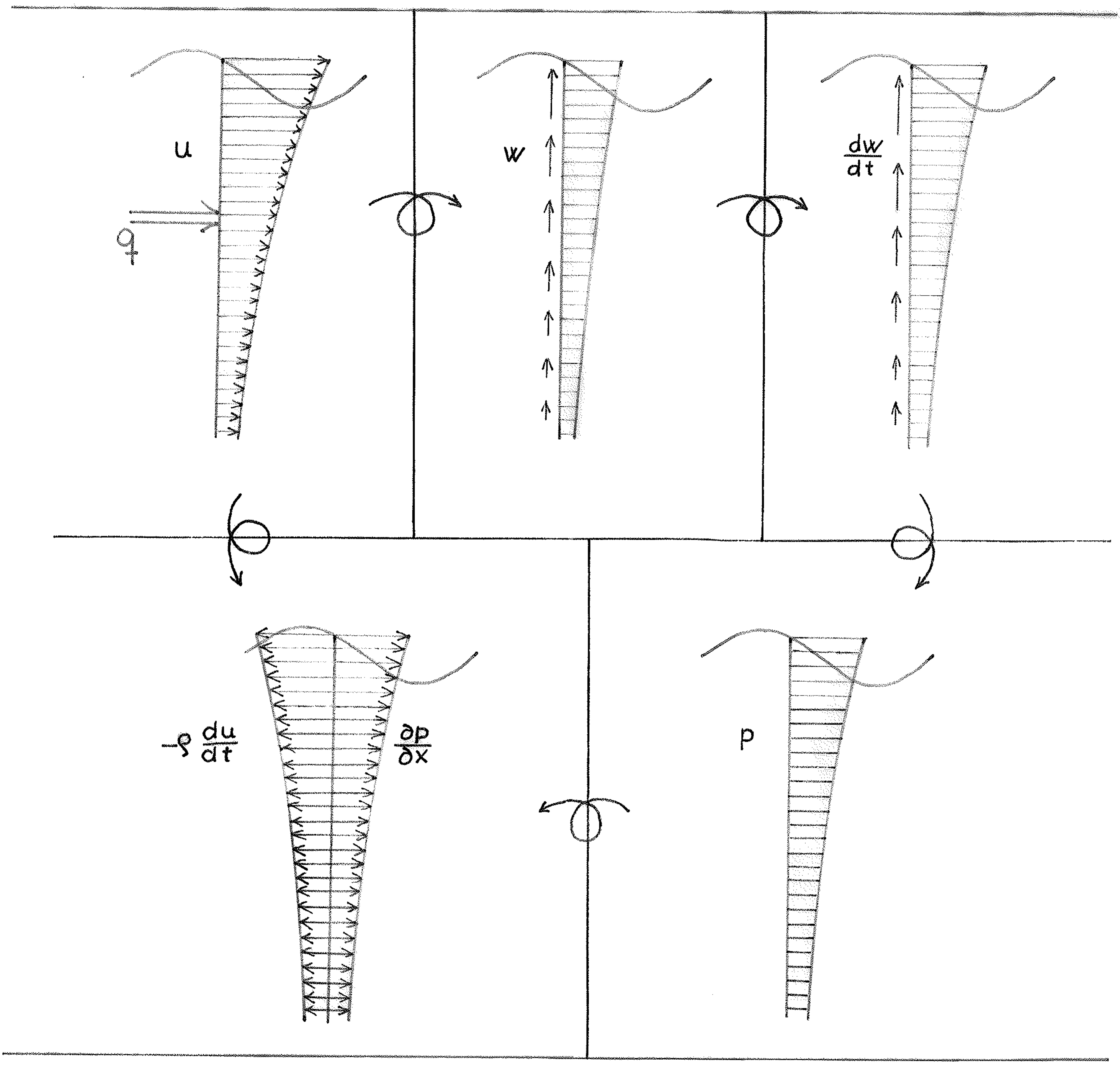
the hydrostatic pressure, which means that the vertical acceleration of the crest above is neglected. So the pressure is not so big, eq. 61 gives 4.1 m.

At the surface of the trough, Airy gives

$$\frac{p^+}{\gamma} = - 5.0 \cdot 0.82 = - 4.1 \text{ m} \tag{72}$$

to be compared with eq. 63. Eq. 72 shows that the Airy theory gives a pressure at the surface of 0.9 m instead of $p = 0$.

Fig. 12. THE BASIC PHYSICAL PRINCIPLES OF THE WAVE THEORY.



The horizontal velocity u is written as an unknown function.
The equation of continuity gives the vertical velocity w.
w and u give the vertical acceleration dw/dt.
The vertical equation of momentum then gives the pressure p.
This gives the horizontal pressure gradient $\partial p/\partial x$.
u and w also give the horizontal acceleration du/dt, or the force of inertia $-\varrho du/dt$.
Finally $-\varrho du/dt$ and $\partial p/\partial x$ must ballance each other at any point of the fluid, which determines the unknown function of u and the wave profile.