CHAPTER XII

STANDING CNOIDAL DEEP WATER WAVES

ABSTRACTS

Second order standing waves for infinite deep water are here described by the 'cnoidal' functions in both space and time. The involved elliptic functions are more complicated than for the progressive cnoidal deep water wave, but the expression for the pressure is rather simple. To make the calculations of pressure easier a graph is given.

INTRODUCTION

After having developed the progressive cnoidal waves in chapter VI and chapter IX it can be of interest to consider the standing wave and see if it is possible to use the cnoidal functions here to get better results. In this chapter we will confine the considerations to infinite deep water. This time we will not start from the beginning, but we will use the results of chapter VII. The notation and the coordinate system will be the same as used in chapter VI.

SINUSOIDAL SOLUTIONS

We get the pressure from eq. 11 in chapter VII (called eq.11VII). For deep water, D/L $\rightarrow\infty$, p will be

$$\frac{p}{\gamma} = \eta - z + \frac{1}{g}\Big\{\Big[\frac{\partial^2\eta}{\partial t^2}\frac{1}{R} - \Big(\frac{\partial\eta}{\partial t}\Big)^2 + \frac{\partial q}{\partial t}\frac{\partial\eta}{\partial x} + q\frac{\partial^2\eta}{\partial x\partial t}\Big]\Big[1 - e^{R(z-\eta)}\Big] + \frac{1}{2}\Big[\Big(\frac{\partial q}{\partial x}\Big)^2 - q\frac{\partial^2 q}{\partial x^2}\Big]\Big[1 - e^{2R(z-\eta)}\Big]\Big\} \qquad (1)$$

The deep water wave equation of second order is got from eq. 13VII by D/L $\rightarrow\infty$

$$g\frac{\partial\eta}{\partial x} + \frac{\partial^3\eta}{\partial x\partial t^2}\frac{1}{R}\Big[1 - e^{R(z-\eta)}\Big] + \frac{\partial q}{\partial t}R\,e^{R(z-\eta)} + \frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial t^2}$$
$$+ \Big[-2\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial t^2} - 3\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial x\partial t} + \frac{\partial q}{\partial t}\frac{\partial^2\eta}{\partial x^2} + q\frac{\partial^3\eta}{\partial x^2\partial t}\Big]\Big[1 - e^{R(z-\eta)}\Big]$$
$$- q\frac{\partial\eta}{\partial t}R^2 e^{R(z-\eta)} + \frac{1}{2}\Big[\frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial x^2} - q\frac{\partial^3 q}{\partial x^3}\Big]\Big[1 - e^{2R(z-\eta)}\Big] = 0 \qquad (2)$$

This equation can give both the progressive and the standing wave solutions of first and second order.

The procedure to find the standing solutions is the same as used in chapter VII. The deep water sinusoidal solutions of first and second order will be given without deductions.

The first order solution is

$$\eta = \eta_1 = \frac{H}{2}\cos\omega t \cos kx \qquad (3)$$

$$\omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{L}, \quad R = k \qquad (4)$$

$$\frac{L}{T} = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \qquad (5)$$

$$q = q_1 = \frac{H}{2}\frac{\omega}{k}\sin\omega t \sin kx \qquad (6)$$

where T is the wave period.

The second order solution of the same type as eq. 15VI is

$$\eta = \eta_1 + \eta_{2a} = \frac{H}{2}\cos\omega t \cos kx + \left(\frac{H}{2}\right)^2\frac{k}{4}(1 + \cos 2\omega t)\cos 2kx \qquad (7)$$

$$q = q_1 + q_{2a} = \frac{H}{2}\frac{\omega}{k}\sin\omega t \sin kx + \left(\frac{H}{2}\right)^2\frac{\omega}{4}\sin 2\omega t \sin 2kx \qquad (8)$$

The second order solution of the same type as eq. 17VI is

$$\eta = \frac{H}{2}\cos\theta_t \cos\theta_x + \eta_p + \Delta D \qquad (9)$$

$$\theta_t = \omega t + \frac{H}{4}k\sin\omega t \cos kx \qquad (10)$$

$$\theta_x = kx + \frac{H}{4}k\cos\omega t \sin kx \qquad (11)$$

$$\eta_p = \left(\frac{H}{2}\right)^2\frac{k}{4}\cos 2kx \qquad (12)$$

$$q = \frac{H}{2}\frac{\omega}{k}\sin\theta_t \sin\theta_x \qquad (13)$$

Eqs. 4 and 5 are the same for the second order waves. The solutions given here can be "tested" in the wave equation, eq. 2.

STANDING CNOIDAL DEEP WATER WAVES

The standing cnoidal wave is found in almost the same way as the progressive cnoidal wave. This means that some of the second order terms in the wave equation, eq. 2, are changed with respect to the first order solution, eq. 3. They are not substituted by the first order solution as is the case when the sinusoidal second order solutions, eqs. 7 and 8, are found (compare with eq. 13VI for the progressive wave). Comparing the progressive solution of eq. 17VI with the corresponding standing solution of eqs. 9–13, it is seen that the expression for the standing cnoidal wave is more complicated and more difficult to find than the expression for the progressive cnoidal wave.

Eq. 6 gives

$$\frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial x^2} = \frac{\partial^3 q}{\partial x^3}\, q \qquad (14)$$

which shows that the last term in eq. 2 can be neglected. Eq. 3 gives

$$-\, q\frac{\partial \eta}{\partial t}R^2 = q\frac{\partial^3 \eta}{\partial x^2 \partial t} \qquad (15)$$

from which it is seen that the last term but one in eq. 2 will also disappear together with part of another term. It is possible in this way to change second order terms whenever needed, and even to replace q by η. But the problem is the first order terms that besides η also contain q and are differentiated with respect to both x and t. It is, however, possible to change eq. 2 into two differential equations. As before, eq. 2 can be split into a z-dependent and a z-independent equation, or, if preferred, a z-dependent and a surface equation. From one of the equations, $\partial q/\partial t$ is isolated and used in the other equation. After some calculations with rather long equations, it is possible to obtain the two equations mentioned above (instead of eq. 2)

$$\frac{\partial \eta}{\partial x} + \frac{1}{R^2}\frac{\partial^3 \eta}{\partial x^3} = -3k\eta\frac{\partial \eta}{\partial x} - 3\left(\frac{H}{2}\right)^2 k^2 \sin^2\omega t \cos kx \sin kx + 3\left(\frac{H}{2}\right)^2 k^2 \cos kx \sin kx \qquad (16)$$

$$\frac{\partial \eta}{\partial t} + \frac{1}{gR}\frac{\partial^3 \eta}{\partial t^3} = -3k\eta\frac{\partial \eta}{\partial t} - 3\left(\frac{H}{2}\right)^2 k\,\omega \cos\omega t \sin\omega t \sin^2 kx \qquad (17)$$

These two equations which can be written in many ways must both be fulfilled at the same time. They are here written so that they fit in with the final solution.

Eqs. 16 and 17 can be "tested" in different ways. The second order solution, eq. 7, is a result of the wave equation, eq. 2. So as eqs. 16 and 17 are correct with eq. 7 then eqs. 16 and 17 are a correct second order transformation of eq. 2. (It was possible a "step" before eqs. 16 and 17 to get two equations that could lead to both the progressive and standing cnoidal waves).

From the sinusoidal solutions for the progressive waves it is seen that eq. 17VI resembles the cnoidal wave in eq. 21VI very much. In the same way, the standing wave of eq. 9 has certain points of resemblance to a cnoidal wave, and it is thus reasonable to expect a cnoidal solution of the type

$$\eta_A = 2H[cn^2\frac{2K_t}{T}t - \frac{1}{2}][cn^2\frac{2K_x}{L}x - \frac{1}{2}] \qquad (18)$$

where K_t and m_t are then functions of x, $K_t = K_t(x)$ and $m_t = m_t(x)$, and in the same way $K_x = K_x(t)$, $m_x = m_x(t)$. Here, it must be remembered that this dependence on x or t does not only affect K in the argument but the whole cn-function.

η_A from eq. 18 must now be "tested" in eqs. 16 and 17. As seen from eq. 23VI, it is not a problem to differentiate cn when the elliptic parameter m is a constant. But in eq. 18, the parameters depend on x in the first cn and on t in the last cn.

In the appendix, it is shown how to differentiate cn with a variable parameter. This is used here to differentiate eq. 18

$$\frac{\partial \eta_A}{\partial t} = -\frac{8K_tH}{T}\, cn\frac{2K_t}{T}t\; sn\frac{2K_t}{T}t\; dn\frac{2K_t}{T}t[cn^2\frac{2K_x}{L}x - \frac{1}{2}]$$

$$- H\cos\omega t\,\sin^2 kx\,\frac{1}{2\pi^2}\,\frac{\partial(m_xK_x^2)}{\partial t} \qquad (19)$$

The last term is a minor term so the sinusoidal approximation was used

$$2H[cn^2\frac{2K_t}{T}t - \frac{1}{2}] = H\cos\omega t \qquad (20)$$

Further differentiation in the same way will give

$$\frac{\partial^3\eta_A}{\partial t^3} = \frac{128\, K_t^3 H}{T^3}\ \mathrm{cn}\frac{2K_t}{T}t\ \mathrm{sn}\frac{2K_t}{T}t\ \mathrm{dn}\frac{2K_t}{T}t\,(1-2m+3m\ \mathrm{cn}^2\frac{2K_t}{T}t)$$

$$\times\,[\mathrm{cn}^2\frac{2K_x}{L}x - \tfrac{1}{2}] + 3H\omega^2\cos\omega t\,\sin^2 kx\frac{1}{2\pi^2}\frac{\partial(m_x K_x^2)}{\partial t}$$

$$+\,3H\omega\sin\omega t\,\sin^2 kx\frac{1}{2\pi^2}\frac{\partial^2(m_x K_x^2)}{\partial t^2}$$

$$-\,H\cos\omega t\,\sin^2 kx\frac{1}{2\pi^2}\frac{\partial^3(m_x K_x^2)}{\partial t^3} \qquad (21)$$

Replacing t by x and T by L gives $\partial\eta_A/\partial x$ and $\partial^3\eta_A/\partial x^3$. Eqs. 18, 19 and 21 are now substituted into eq. 17, and the corresponding equations are substituted into eq. 16. Comparing eq. 16 for the standing wave with eq. 20VI for the progressive wave it is seen that they are very much alike. The solution is found in exactly the same way. It has previously been explained that it was acceptable to use $R = 2\pi/L$ instead of eq. 26VI. In the same way, gR can be substituted by the second order sinusoidal value $gR = \omega^2 = (2\pi/T)^2$. These approximations can be shown to be correct in a second order theory.

Corresponding to eq. 25VI it will then be found that

$$m_t K_t^2 = \frac{1}{2}\pi^3\frac{H}{L}\cos kx = \frac{1}{4}\pi^2 H k\cos kx \qquad (22)$$

$$m_x K_x^2 = \frac{1}{2}\pi^3\frac{H}{L}\cos\omega t = \frac{1}{4}\pi^2 H k\cos\omega t \qquad (23)$$

η_A of eq. 18 can, however, not alone completely fulfil eqs. 16 and 17. As in eq. 9, there will be a permanent standing wave of

$$\eta_p = (\frac{H}{2})^2\frac{k}{4}\cos 2kx \qquad (24)$$

The final solution is then

$$\eta = \eta_A + \eta_p + \Delta D \qquad (25)$$

where ΔD is found by demanding $\int_0^L \eta\, dx = 0$.

η in eq. 25 could in a second order theory be written as one cnoidal term as for the progressive wave, eq. 21VI, but then the mathematical simplicity of eqs. 22 and 23 would be lost.

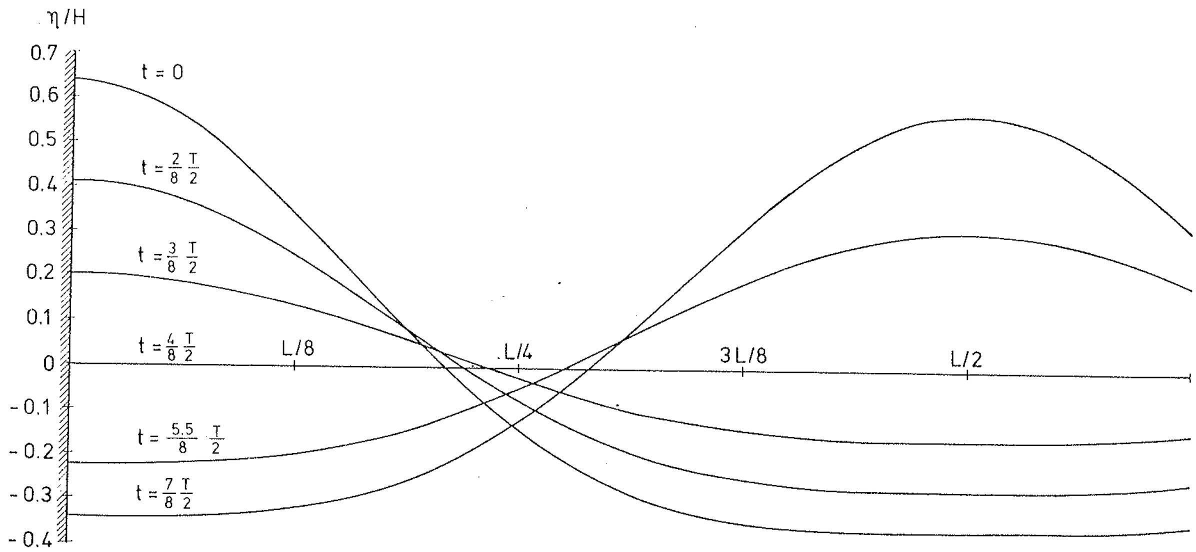


Fig. 1. The surface profile of the standing cnoidal wave at different times. t = 0 is for the maximum crest at the wall. The profile for t = (7/8)T/2 corresponds to the profile for t = (1/8)T/2, etc. H_S/L = 18%.

PRESSURE

The particle velocities, u and w, and the pressure, p, are determined by the equations which were found at the beginning in chapter VII together with the solutions for η for the progressive or the standing wave. The expression for u is simple, but the expression for p, eq. 1, contains several terms. In all the second order terms it is permissible to use the first order expressions for η and q.

Eq. 1 can be written more simply. The wave pressure, i.e. pressure above hydrostatic pressure from still water, is called p_b^+ for $z \to -\infty$, so that

$$\frac{p_b^+}{\gamma} = \frac{p}{\gamma} + z \qquad \text{for } z \to -\infty \tag{26}$$

For a progressive wave, it can easily be seen that $p_b^+ = 0$. For a standing wave, p_b^+ can be found by considering the vertical oscillations of the centre of mass of a wave length of water. Then the pressure correct to second order is found by considering a first order wave

$$\frac{p_b^+}{\gamma} = -\frac{\pi}{4} H \frac{H}{L} \cos 2\,\omega t \qquad (27)$$

Eq. 26 is used in eq. 1 together with eq. 1VI to get

$$\frac{p_b^+}{\gamma} - \eta = \frac{1}{g}\left\{\frac{\partial^2\eta}{\partial t^2}\,\frac{1}{R} - \frac{1}{2}\left(\frac{\partial\eta}{\partial t}\right)^2 + \frac{\partial q}{\partial t}\,\frac{\partial\eta}{\partial x} + \frac{3}{2}\,q\,\frac{\partial^2\eta}{\partial x\partial t}\right\} \qquad (28)$$

By this, eq. 1 can be written as

$$\frac{p}{\gamma} + z = \eta\, e^{R(z-\eta)} + \frac{1}{2g}\left[\left(\frac{\partial q}{\partial x}\right)^2 - q\frac{\partial^2 q}{\partial x^2}\right]\left[e^{R(z-\eta)} - e^{2R(z-\eta)}\right] + \frac{p_b^+}{\gamma}\left[1 - e^{R(z-\eta)}\right] \qquad (29)$$

Using eqs. 6 and 27 in the second order terms, eq. 29 can be written as given in the review of cnoidal formulas in the appendix.

Eq. 29 can also be found directly from eq. 1 by rewriting the terms to other second order terms in the same way as in eqs. 14, 15 . Eqs. 1 and 29 will by numerical examples give small differences in the pressure, but these differences are of third order magnitude, although such differences can be felt rather important.

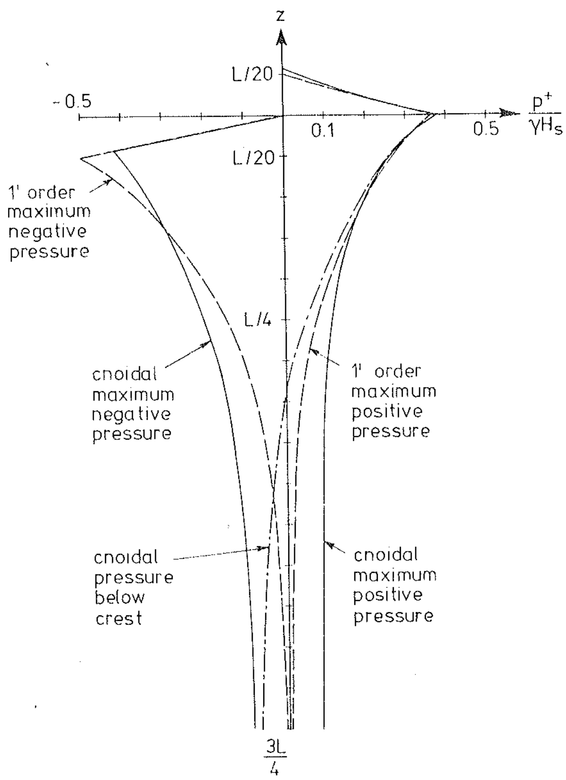


Fig. 2. Pressure on the vertical wall from a standing wave with the steepness $H_s/L = 10\%$. Comparison of the 1' order sinusoidal theory with the second order cnoidal theory.

APPENDIX I

DIFFERENTIATION OF JACOBIAN ELLIPTIC FUNCTION WITH VARIABLE PARAMETER

The Jacobian elliptic functions are defined as

$$\mathrm{cn}\,\frac{2K}{L}x = \cos(\mathrm{am}\,\frac{2K}{L}x) \tag{30}$$

where am is called the amplitude.

The Fourier expansion of am is

$$\mathrm{am}\,\frac{2K}{L}x = \frac{\pi}{L}x + \mathrm{sech}\,\pi\frac{K_c}{K}\sin\frac{2\pi}{L}x + \cdots\cdots \tag{31}$$

where $K = K(m)$ and $K_c = K(m_c)$ with $m_c = 1 - m$.

The first term is seen to be the usual argument of $\cos^2$ in a sinusoidal description. So the second term can be thought of as a small term to give a second order correction, which is reasonable as long as K_c is as large as K_c will be in the possible solutions here. This means that the second term can be approximated with sinusoidal expressions as it was done earlier with second order terms in the wave equation. So, in this term m is regarded small. Then K and K_c can be approximated by the first term of their expansion in series

$$K \simeq \frac{\pi}{2} \quad ; \quad K_c \propto \ln\frac{4}{\sqrt{m}} \tag{32}$$

Now, sech can be approximated as

$$\mathrm{sech}\,\pi\frac{K_c}{K} \simeq 2\exp(-\pi\frac{K_c}{K}) \simeq 2\exp(-2\ln\frac{4}{\sqrt{m}}) = \frac{m}{8} \tag{33}$$

which is differentiated to give

$$\frac{\partial\,\mathrm{sech}\,\pi\frac{K_c}{K}}{\partial t} \simeq \frac{1}{8}\frac{\partial m}{\partial t} \simeq \frac{1}{2\pi^2}\frac{\partial(mK^2)}{\partial t} \tag{34}$$

where eq. 32 was used.

By means of the usual rules of differentiation of Jacobian elliptic functions, eqs. 30, 31 and 34 yield

$$\frac{\partial\,\mathrm{cn}^2\frac{2K}{L}x}{\partial t} = -2\,\mathrm{cn}\frac{2K}{L}x\,\mathrm{sn}\frac{2K}{L}x\frac{\partial\,\mathrm{am}\frac{2K}{L}x}{\partial t}$$

$$\simeq -\frac{1}{\pi^2}\,\mathrm{cn}\frac{2K}{L}x\,\mathrm{sn}\frac{2K}{L}x\,\frac{\partial(mK^2)}{\partial t}\sin\frac{2\pi}{L}x \tag{35}$$

In second order terms, the approximation can be used

$$2\ \mathrm{cn}\,\frac{2K}{L}\,x\ \mathrm{sn}\,\frac{2K}{L}\,x \simeq 2\cos\frac{\pi}{L}x\,\sin\frac{\pi}{L}x = \sin\frac{2\pi}{L}x \tag{36}$$

and now eq. 35 will be

$$\frac{\partial\ \mathrm{cn}^2\,\frac{2K}{L}x}{\partial t} = -\frac{1}{2\pi^2}\sin^2\frac{2\pi}{L}x\,\frac{\partial(mK^2)}{\partial t} \tag{37}$$

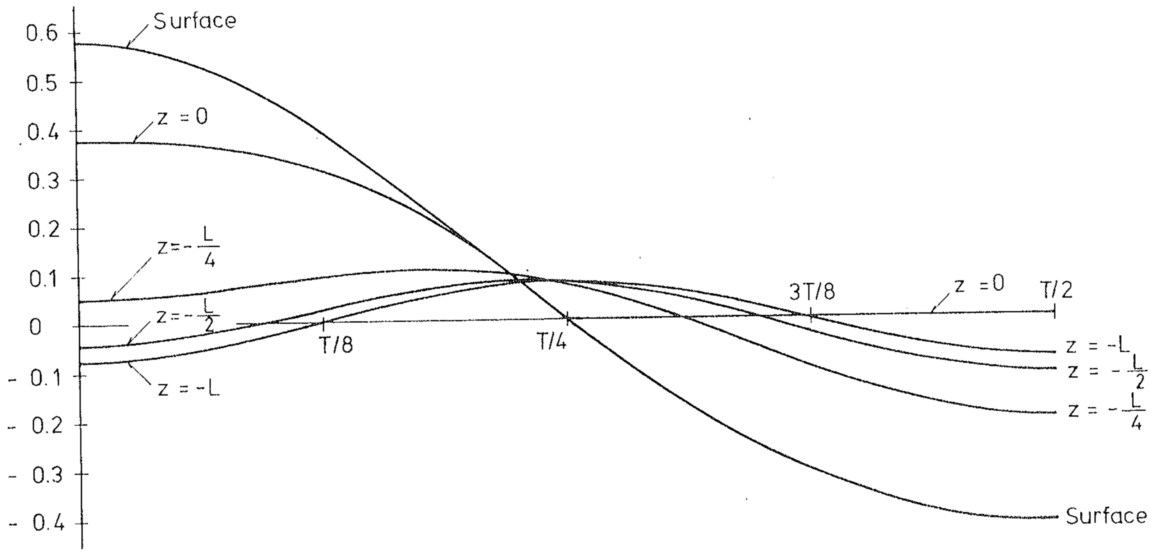


Fig. 3. The pressure as a function of time from the standing cnoidal wave on a vertical wall at different depths below mean water level. The steepness is the same as in fig. 2: $H_S/L = 10\%$.

APPENDIX II

REVIEW OF CNOIDAL FORMULAS

$$\eta = \eta_A + \eta_p + \Delta D \tag{38}$$

$$\eta_A = 2H[cn^2 \frac{2K_t}{T} t - \tfrac{1}{2}][cn^2 \frac{2K_x}{L} x - \tfrac{1}{2}] \tag{39}$$

$$m_t K_t^2 = \tfrac{1}{2}\pi^3 \frac{H}{L} \cos kx \tag{40}$$

$$m_x K_x^2 = \tfrac{1}{2}\pi^3 \frac{H}{L} \cos \omega t \tag{41}$$

$$k = \frac{2\pi}{L} \; ; \; \omega = \frac{2\pi}{T} \tag{42}$$

$$(c =) \; \frac{L}{T} = \sqrt{\frac{g}{k}} \tag{43}$$

$$\eta_p = (\frac{H}{2})^2 \frac{k}{4} \cos 2kx \tag{44}$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \eta_A}{\partial t} = - \frac{\partial q}{\partial x} \text{ is given by eq. 19} \tag{45}$$

$$u = q k e^{k(z-\eta)} \tag{46}$$

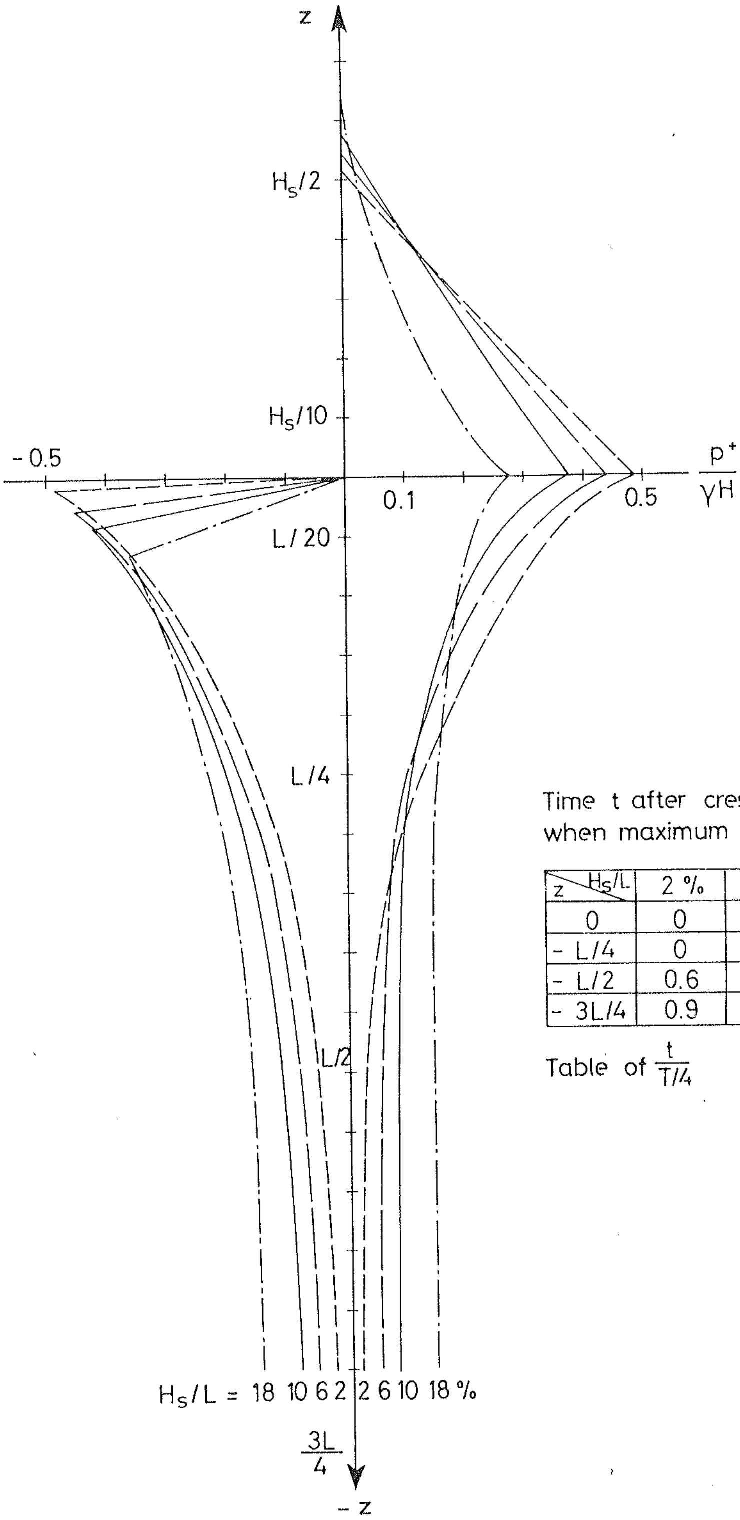
$$w = [\frac{\partial \eta}{\partial t} + q \frac{\partial \eta}{\partial x}] e^{k(z-\eta)} \tag{47}$$

$$\frac{p}{\gamma} + z = \eta e^{k(z-\eta)} + \frac{\pi}{4} H \frac{H}{L} \sin^2 \omega t [e^{k(z-\eta)} - e^{2k(z-\eta)}] - \frac{\pi}{4} H \frac{H}{L} \cos 2\omega t [1 - e^{k(z-\eta)}] \tag{48}$$

H_s/L	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
$\Delta D/H_s$	0.008	0.016	0.023	0.031	0.039	0.047	0.054	0.061	0.069	0.076
η_{cm}/H_s	0.52	0.53	0.55	0.56	0.58	0.59	0.61	0.62	0.64	0.65

H_s is the wave height of the standing wave

η_{cm} is the maximum crest height by the wall.



Time t after crest
when maximum pressure

z \ H_s/L	2 %	6 %	10 %	18 %
0	0	0	0	0.5
- L/4	0	0.5	0.7	0.8
- L/2	0.6	0.9	0.9	1.0
- 3L/4	0.9	1.0	1.0	1.0

Table of $\frac{t}{T/4}$

Fig. 4 . Maximum positive and maximum negative wave pressure at a vertical wall below the standing cnoidal wave for different steepness H_s/L = 2%, 6%, 10%, and 18%. Note that the unit is different below and above the mean water level, z = 0. As shown in fig. 3, the maximum positive pressure will not necessarily occur when having a crest, but at the time given in the table.