# CHAPTER XI

## PROGRESSIVE THIRD ORDER SINUSOIDAL WAVE

ABSTRACTS

The theory of the second order sinusoidal wave of chapter VII is here continued to give the third order theory. The development of the theory may be of hydrodynamic interest because it involves a little more than just including more terms. The results are of interest mainly by showing the celerity to increase with increasing wave height.

### INTRODUCTION

In chapter VI we found that for deep water waves the second order theory just like the first order theory had the solution $R = k$. So to find the third order theory was just a matter of including more terms. In chapter VII we found that for second order waves on arbitrary depth we would for one of the second harmonic waves get $R = k$. This makes the convective terms in the expressions for accelerations more complicated.

The chapter here is a copy of the author's thesis of 1971.

In chapter VII the sinusoidal theory for progressive and standing waves of first and second order was developed. The procedure was to use the equation of continuity on the water discharge, $Q$, through a vertical to find the surface elevation, $\eta$:

$$\frac{\partial Q}{\partial x} = -\frac{\partial \eta}{\partial t} \tag{1}$$

where $x$ is the horizontal co-ordinate and $t$ the time. The horizontal particle velocity, $u$, was written :

$$u = \sum_{m=1}^{\infty} Q_m \cdot \frac{R_m \cdot \cosh(R_m \cdot z)}{\sinh(R_m \cdot (D+\eta))} \tag{2}$$

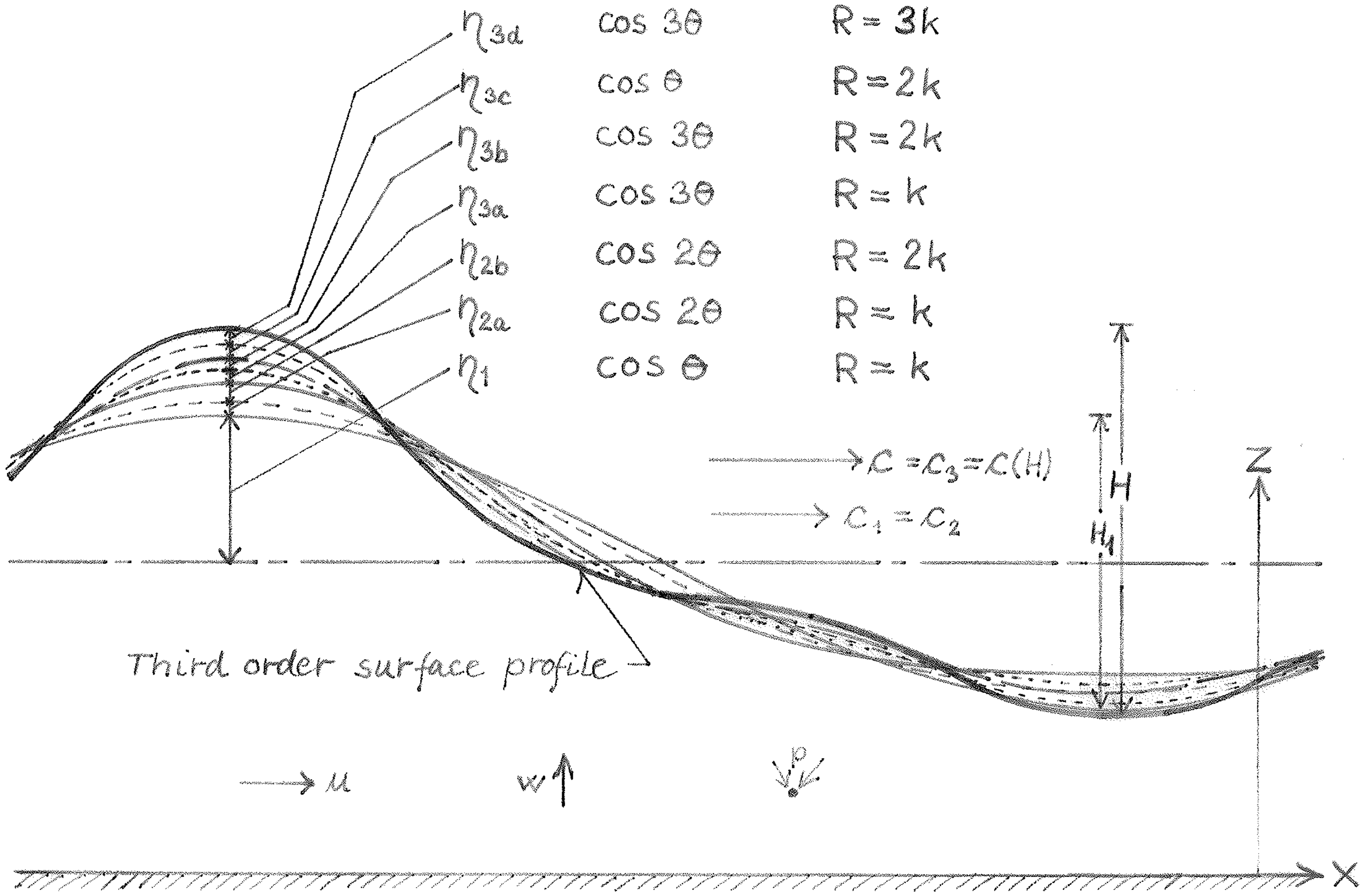


Fig. 1. The result of the third order theory will be as shown in this symbolical drawing. For $\eta$ we get $\eta = \eta_1+\eta_{2a}+\eta_{2b}+\eta_{3a}+\eta_{3b}+\eta_{3c}+\eta_{3d}$ where $\eta_1$ was found in chapter IV and $\eta_{2a}$ and $\eta_{2b}$ were found in chapter VII. $\eta_{3a}$, $\eta_{3b}$, $\eta_{3c}$ and $\eta_{3d}$ are found in this chapter in eqs. 26, 27, 28, and 29. These four third order components of $\eta$ are not combined in one term because they have different R-values. This is used for u, w, and p. E.g. for u we have $u = u_1+u_{2a}+u_{2b}+u_{3a}+u_{3b}+u_{3c}+u_{3d}$ where $u_i = c\eta_i R_i \cosh R_i z/\sinh R_i(D+\eta)$, so that e.g. we for $u_{2b}$ get $u_{2b} = c\eta_{2b}\, 2k \cosh 2kz/\sinh 2k(D+\eta)$. $u_{2b}$ is then used all the way from the bottom, z = o, to the surface, $z = D+\eta$. The third order celerity depends on the wave height, eq. 33, and on the rotation of second order (not considered in this chapter). The wave height, $H_1$, of the first order component will be less than H, eq. 36, which means that this figure cannot be used directly to compare the results of the first and second order wave with the third order wave.

where m is an integer, z the vertical co-ordinate, D the mean water depth and, with L being the wave-length, $R_m = 2 \cdot \pi / L$ in the first order theory with m having only one value.

Through the equation of continuity the vertical particle velocity, w was then found to, omitting $\Sigma$ :

$$w = -\frac{\partial Q}{\partial x} \cdot \frac{\sinh(R \cdot z)}{\sinh(R \cdot (D+\eta))}$$

$$+ Q \cdot \frac{R \cdot \cosh(R \cdot (D+\eta)) \cdot \sinh(R \cdot z)}{\sinh^2(R \cdot (D+\eta))} \cdot \frac{\partial \eta}{\partial x} \qquad (3)$$

By the vertical and horizontal dynamic equation a wave equation was produced, and from that the solutions of first and second order were found. For the progressive wave the surface elevation, $\eta$, consisted of one first order term, $\eta$, and two second order terms $\eta_{2a}$ and $\eta_{2b}$ :

$$\eta = \eta_1 + \eta_{2a} + \eta_{2b} = \frac{H}{2} \cdot \cos(\omega \cdot t - k \cdot x)$$

$$+ \left(\frac{H}{2}\right)^2 \cdot \frac{k}{2} \cdot \coth(k \cdot D) \cdot \cos(2 \cdot \omega \cdot t - 2 \cdot k \cdot x)$$

$$+ \left(\frac{H}{2}\right)^2 \cdot \frac{3}{4} \cdot k \cdot \frac{\coth(k \cdot D)}{\sinh^2(k \cdot D)} \cdot \cos(2 \cdot \omega \cdot t - 2 \cdot k \cdot x) \qquad (4)$$

where H is the wave-height, $k = 2 \cdot \pi / L$, and $\omega = 2 \cdot \pi / T$ with T being the wave-period. When using (2) and (3) and the formulæ made from (2) and (3) R must be $R = k$ for $\eta_1$ and $\eta_{2a}$ and $R = 2 \cdot k$ for $\eta_{2b}$, whereby the dynamic expressions differ important from the usually used. To give a better understanding of the procedure the process will be expanded to include the progressive wave of third order.

The third order terms that make up the third order equation consist of $\eta_1$, $Q_1$ alone or combined with $\eta_{2a}$, $Q_{2a}$ or $\eta_{2b}$, $Q_{2b}$ which are all known, or consist of $\eta_3$, $Q_3$ which are to be found. As R is different for $Q_1$ and $Q_{2b}$ it is important here to distinguis which Q an R belongs to in terms with double Q like the convective terms. Where doubt is possible the belonging is indicated with index $_a$ and index $_b$. In this way the vertical acceleration $G_z$ will be:

$$G_z = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial z} \cdot w + \frac{\partial w}{\partial x} \cdot u = - \frac{\partial^2 Q}{\partial x \cdot \partial t} \cdot \frac{\sinh(R \cdot z)}{\sinh(R \cdot (D+\eta))}$$

$$+ \left[\frac{\partial Q}{\partial x} \cdot \frac{\partial \eta}{\partial t} + \frac{\partial Q}{\partial t} \cdot \frac{\partial \eta}{\partial x} + Q \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t}\right] \cdot$$

$$\frac{R \cdot \cosh(R \cdot (D+\eta)) \cdot \sinh(R \cdot z)}{\sinh^2(R \cdot (D+\eta))}$$

$$- Q \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial \eta}{\partial t} \cdot \frac{R^2 \cdot \sinh(R \cdot z)}{\sinh^3(R \cdot (D+\eta))} \cdot [\cosh^2(R \cdot (D+\eta)) + 1]$$

$$+ \left[\frac{\partial Q_a}{\partial x} \cdot \frac{\partial Q_b}{\partial x} - Q_a \cdot \frac{\partial^2 Q_b}{\partial x^2}\right] \cdot \frac{R_a \cdot \cosh(R_a \cdot z)}{\sinh(R_a \cdot (D+\eta))} \cdot \frac{\sinh(R_b \cdot z)}{\sinh(R_b \cdot (D+\eta))}$$

$$- \frac{\partial Q_a}{\partial x} \cdot \frac{R_a \cdot \cosh(R_a \cdot z)}{\sinh(R_a \cdot (D+\eta))} \cdot Q_b \cdot \frac{\partial \eta}{\partial x}$$

$$\cdot \frac{R_b \cdot \cosh(R_b \cdot (D+\eta)) \cdot \sinh(R_b \cdot z)}{\sinh^2(R_b \cdot (D+\eta))}$$

$$- \frac{\partial Q_b}{\partial x} \cdot \frac{\sinh(R_b \cdot z)}{\sinh(R_b \cdot (D+\eta))} \cdot Q_a \cdot \frac{\partial \eta}{\partial x} \cdot$$

$$\frac{R_a^2 \cdot \cosh(R_a \cdot (D+\eta)) \cdot \cosh(R_a \cdot z)}{\sinh^2(R_a (D+\eta))}$$

$$+ Q_a \cdot \frac{\partial \eta}{\partial x} \cdot \frac{R_a^2 \cdot \cosh(R_a \cdot (D+\eta)) \cdot \cosh(R_a \cdot z)}{\sinh^2(R_a \cdot (D+\eta))} \cdot Q_b \cdot \frac{\partial \eta}{\partial x}$$

$$\cdot \frac{R_b \cdot \cosh(R_b\cdot(D+\eta)) \cdot \sinh(R_b\cdot z)}{\sinh^2(R_b\cdot(D+\eta))}$$

$$+\left[2 \cdot \frac{\partial Q_b}{\partial x} \cdot \frac{\partial \eta}{\partial x} + Q_b \cdot \frac{\partial^2\eta}{\partial x^2}\right] \cdot \frac{R_b \cdot \cosh(R_b\cdot(D+\eta)) \cdot \sinh(R_b\cdot z)}{\sinh^2(R_b\cdot(D+\eta))}$$

$$\cdot Q_a \cdot \frac{R_a \cdot \cosh(R_a\cdot z)}{\sinh(R_a\cdot(D+\eta))} - Q_b \cdot \left(\frac{\partial\eta}{\partial x}\right)^2 \cdot \frac{R_b^{\,2} \cdot \sinh(R_b\cdot z)}{\sinh^3(R_b\cdot(D+\eta))}$$

$$\cdot \left[\cosh^2(R_b\cdot(D+\eta)) + 1\right] \cdot Q_a \cdot \frac{R_a \cdot \cosh(R_a\cdot z)}{\sinh(R_a\cdot(D+\eta)} \tag{5}$$

The vertical dynamic equation is:

$$-\frac{\partial p}{\partial z} = \gamma + \frac{\gamma}{g} \cdot G_z \tag{6}$$

where $\gamma$ is the unit weight of the water and $g$ the acceleration of gravity. The pressure $p$ is then found by integration of (5) and (6). It is necessary to distinguish between the case of $R_a = R_b$ and $R_a \neq R_b$. This will be done by the term $A_1(\eta,z)$:

$$\frac{p}{\gamma} = D + \eta - z + \frac{1}{g} \cdot \left\{- \frac{\partial^2 Q}{\partial x\cdot\partial t} \cdot \frac{\cosh(R\cdot(D+\eta)) - \cosh(R\cdot z)}{R \cdot \sinh(R\cdot(D+\eta))}\right.$$

$$+ \left[\frac{\partial Q}{\partial x} \cdot \frac{\partial\eta}{\partial t} + \frac{\partial Q}{\partial t} \cdot \frac{\partial\eta}{\partial x} + Q \cdot \frac{\partial^2\eta}{\partial x\cdot\partial t}\right] \cdot$$

$$\frac{\cosh(R\cdot(D+\eta)) \cdot [\cosh(R\cdot(D+\eta)) - \cosh(R\cdot z)]}{\sinh^2(R\cdot(D+\eta))}$$

$$- Q \cdot \frac{\partial\eta}{\partial x} \cdot \frac{\partial\eta}{\partial t} \cdot \frac{R \cdot [\cosh(R\cdot(D+\eta)) - \cosh(R\cdot z)]}{\sinh^3(R\cdot(D+\eta))} \cdot$$

$$[\cosh^2(R\cdot(D+\eta)) + 1]$$

$$+ \left\{\left[\frac{\partial Q_a}{\partial x} \cdot \frac{\partial Q_b}{\partial x} - Q_a \cdot \frac{\partial^2 Q_b}{\partial x^2}\right] \cdot \frac{R_a}{\sinh(R_a\cdot(D+\eta)) \cdot \sinh(R_b\cdot(D+\eta))}\right.$$

$$- \frac{\partial Q_a}{\partial x} \cdot \frac{R_a}{\sinh(R_a \cdot (D+\eta))} \cdot Q_b \cdot \frac{\partial \eta}{\partial x} \cdot \frac{R_b \cdot \cosh(R_b \cdot (D+\eta))}{\sinh^2(R_b \cdot (D+\eta))}$$

$$- \frac{\partial Q_b}{\partial x} \cdot \frac{1}{\sinh(R_b \cdot (D+\eta))} \cdot Q_a \cdot \frac{\partial \eta}{\partial x} \cdot \frac{R_a^{\,2} \cdot \cosh(R_a \cdot (D+\eta))}{\sinh^2(R_a \cdot (D+\eta))}$$

$$+ Q_a \cdot \frac{\partial \eta}{\partial x} \cdot \frac{R_a^{\,2} \cdot \cosh(R_a \cdot (D+\eta))}{\sinh^2(R_a \cdot (D+\eta))}$$

$$\cdot Q_b \cdot \frac{\partial \eta}{\partial x} \cdot \frac{R_b \cdot \cosh(R_b \cdot (D+\eta))}{\sinh^2(R_b \cdot (D+\eta))}$$

$$+ \left[2 \cdot \frac{\partial Q_b}{\partial x} \cdot \frac{\partial \eta}{\partial x} + Q_b \cdot \frac{\partial^2 \eta}{\partial x^2}\right] \cdot \frac{R_b \cdot \cosh(R_b \cdot (D+\eta))}{\sinh^2(R_b \cdot (D+\eta))}$$

$$\cdot Q_a \cdot \frac{R_a}{\sinh(R_a \cdot (D+\eta))} - Q_b \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 \cdot \frac{R_b^{\,2}}{\sinh^3(R_b \cdot (D+\eta))}$$

$$\cdot \left[\cosh^2(R_b \cdot (D+\eta)) + 1\right] \cdot Q_a \cdot \frac{R_a}{\sinh(R_a \cdot (D+\eta))}\Big\} \cdot A_1(\eta, z)\Big\} \quad (7)$$

where:

$$A_1(\eta, z) = \frac{1}{4R} \cdot [\cosh(2 \cdot R \cdot (D+\eta) - \cosh(2 \cdot R \cdot z)] \quad (8)$$

for $R_a = R_b = R_1$, and:

$$A_1(\eta, z) = \frac{\cosh((R_b+R_a) \cdot (D+\eta)) - \cosh((R_b+R_a) \cdot z)}{2 \cdot (R_b+R_a)}$$

$$+ \frac{\cosh((R_b - R_a) \cdot (D+\eta)) - \cosh((R_b - R_a) \cdot z)}{2 \cdot (R_b - R_a)} \quad (9)$$

for $R_a \neq R_b$

(7) is differentiated and the term depending on $R_a$ and $R_b$ is called $A_2(\eta, z)$. Terms of no more than third order are taken along:

$$\frac{g}{\gamma} \cdot \frac{\partial p}{\partial x} = g \cdot \frac{\partial \eta}{\partial x} - \frac{\partial^3 Q}{\partial x^2 \cdot \partial t} \cdot \frac{\cosh(R \cdot (D+\eta)) - \cosh(R \cdot z)}{R \cdot \sinh(R \cdot (D+\eta))}$$

$$- \frac{\partial^2 Q}{\partial x \cdot \partial t} \cdot \frac{\partial \eta}{\partial x} + \left[2 \cdot \frac{\partial^2 Q}{\partial x \cdot \partial t} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial^2 Q}{\partial x^2} \cdot \frac{\partial \eta}{\partial t} + 2 \cdot \frac{\partial Q}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t}\right.$$

$$\left. + \frac{\partial Q}{\partial t} \cdot \frac{\partial^2 \eta}{\partial x^2} + Q \cdot \frac{\partial^3 \eta}{\partial x^2 \cdot \partial t}\right] \cdot$$

$$\frac{\cosh(R \cdot (D+\eta)) \cdot [\cosh(R \cdot (D+\eta)) - \cosh(R \cdot z)]}{\sinh^2(R \cdot (D+\eta))}$$

$$+ \left[\frac{\partial Q}{\partial x} \cdot \frac{\partial \eta}{\partial t} + \frac{\partial Q}{\partial t} \cdot \frac{\partial \eta}{\partial x} + Q \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t}\right] \cdot \frac{\partial \eta}{\partial x} \cdot$$

$$\left[- \frac{R \cdot [\cosh^2(R \cdot (D+\eta)) + 1] \cdot [\cosh(R \cdot (D+\eta)) - \cosh(R \cdot z)]}{\sinh^3(R \cdot (D+\eta))}\right.$$

$$\left. + R \cdot \coth(R \cdot (D+\eta))\right] - \left[\frac{\partial Q}{\partial x} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial \eta}{\partial t}\right.$$

$$\left. + Q \cdot \frac{\partial^2 \eta}{\partial x^2} \cdot \frac{\partial \eta}{\partial t} + Q \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t}\right] \cdot$$

$$\frac{R \cdot [\cosh(R \cdot (D+\eta)) - \cosh(R \cdot z)]}{\sinh^3(R \cdot (D+\eta))} \cdot [\cosh^2(R \cdot (D+\eta)) + 1)]$$

$$+ A_2(\eta, z) \qquad (10)$$

where for $R_a = R_b = R$

$$A_2(\eta, z) = \left[\frac{\partial Q}{\partial x} \cdot \frac{\partial^2 Q}{\partial x^2} - Q \cdot \frac{\partial^3 Q}{\partial x^3}\right] \cdot$$

$$\frac{\cosh(2 \cdot R \cdot (D+\eta)) - \cosh(2 \cdot R \cdot z)}{4 \cdot \sinh^2(R \cdot (D+\eta))}$$

$$+ \left[\left(\frac{\partial Q}{\partial x}\right)^2 - Q \cdot \frac{\partial^2 Q}{\partial x^2}\right] \cdot \frac{\partial \eta}{\partial x} \cdot$$

$$\left[- \frac{R \cdot \cosh(R \cdot (D+\eta)) \cdot [\cosh(2 \cdot R \cdot (D+\eta)) - \cosh(2 \cdot R \cdot z)]}{2 \cdot \sinh^3(R \cdot (D+\eta))}\right.$$

$$+ R \cdot \coth(R\cdot(D+\eta))\Big] + \left[2 \cdot Q \cdot \frac{\partial Q}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x^2} + Q^2 \cdot \frac{\partial^3 \eta}{\partial x^3}\right] \cdot$$

$$\frac{R \cdot \cosh(R\cdot(D+\eta)) \cdot [\cosh(2\cdot R\cdot(D+\eta)) - \cosh(2\cdot R\cdot z)]}{4 \cdot \sinh^3(R\cdot(D+\eta))} \qquad (11)$$

For $R_a \neq R_b$ there is only needed the term of second order in $A_2(\eta, z)$:

$$A_2(\eta,z) = \left[\frac{\partial^2 Q_a}{\partial x^2} \cdot \frac{\partial Q_b}{\partial x} - Q_a \cdot \frac{\partial^3 Q_b}{\partial x^3}\right] \cdot$$

$$\frac{R_a}{\sinh(R_a\cdot(D+\eta)) \cdot \sinh(R_b\cdot(D+\eta))} \cdot$$

$$\left[\frac{\cosh((R_b+R_a)\cdot(D+\eta)) - \cosh((R_b+R_a)\cdot z)}{2 \cdot (R_b+R_a)}\right.$$

$$\left. + \frac{\cosh((R_b-R_a)\cdot(D+\eta)) - \cosh((R_b-R_a)\cdot z)}{2 \cdot (R_b-R_a)}\right] \qquad (12)$$

The general term in (2) gives the horizontal acceleration, $G_x$:

$$G_x = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot u + \frac{\partial u}{\partial z} \cdot w = \frac{\partial Q}{\partial t} \cdot \frac{R \cdot \cosh(R\cdot z)}{\sinh(R\cdot(D+\eta))}$$

$$- Q \cdot \frac{\partial \eta}{\partial t} \cdot \frac{R^2 \cdot \cosh(R\cdot(D+\eta)) \cdot \cosh(R\cdot z)}{\sinh^2(R\cdot(D+\eta))}$$

$$+ \frac{\partial Q_a}{\partial x} \cdot \frac{R_a \cdot \cosh(R_a\cdot z)}{\sinh(R_a\cdot(D+\eta))} \cdot Q_b \cdot \frac{R_b \cdot \cosh(R_b\cdot z)}{\sinh(R_b\cdot(D+\eta))}$$

$$- Q_a \cdot \frac{\partial \eta}{\partial x} \cdot \frac{R_a^2 \cdot \cosh(R_a\cdot(D+\eta)) \cdot \cosh(R_a\cdot z)}{\sinh^2(R_a\cdot(D+\eta))} \cdot$$

$$Q_b \cdot \frac{R_b \cdot \cosh(R_b\cdot z)}{\sinh(R_b\cdot(D+\eta))}$$

$$- Q_a \cdot \frac{R_a^{\;2} \cdot \sinh(R_a \cdot z)}{\sinh(R_a \cdot (D+\eta))} \cdot \frac{\partial Q_b}{\partial x} \cdot \frac{\sinh(R_b \cdot z)}{\sinh(R_b \cdot (D+\eta))}$$

$$+ Q_a \cdot \frac{R_a^{\;2} \cdot \sinh(R_a \cdot z)}{\sinh(R_a \cdot (D+\eta))} \cdot Q_b \cdot$$

$$\frac{R_b \cdot \cosh(R_b \cdot (D+\eta)) \cdot \sinh(R_b \cdot z)}{\sinh^2(R_b \cdot (D+\eta))} \cdot \frac{\partial \eta}{\partial x} \qquad (13)$$

To give the wave-equation (10) and (13) are collected in the horizontal dynamic equation:

$$G_x = - \frac{g}{\gamma} \cdot \frac{\partial p}{\partial x} \qquad (14)$$

This wave-equation is then divided into a z-dependent and a z-independent equation. The z-independent equation of first order was:

$$g \cdot \frac{\partial \eta_1}{\partial x} + \frac{\partial^3 \eta_1}{\partial x \cdot \partial t^2} \cdot \frac{1}{k} \cdot \coth(k \cdot (D+\eta)) = 0 \qquad (15)$$

The z-independent equation of second order was:

$$g \cdot \frac{\partial \eta_2}{\partial x} + \frac{\partial^3 \eta_2}{\partial x \cdot \partial t^2} \cdot \frac{1}{R} \cdot \coth(R \cdot (D+\eta)) =$$

$$(\frac{H}{2})^2 \cdot \omega^2 \cdot k \cdot \cos(\theta) \cdot \sin(\theta) \cdot$$

$$[3 - \coth^2(k \cdot D) - 8 \cdot \coth^2(k \cdot (D+\eta))] \qquad (16)$$

where the substitution was made:

$$\theta = 2 \cdot \omega \cdot t - 2 \cdot k \cdot x \qquad (17)$$

The z-dependent equation of second order was:

$$\frac{\partial Q_{2a}}{\partial t} \cdot \frac{k \cdot \cosh(k \cdot z)}{\sinh(k \cdot (D+\eta))} - \frac{\partial^3 \eta_{2a}}{\partial x \cdot \partial t^2} \cdot \frac{\cosh(k \cdot z)}{k \cdot \sinh(k \cdot (D+\eta))}$$

$$= 3 \cdot (\frac{H}{2})^2 \cdot \omega^2 \cdot k \cdot \sin(2 \cdot \theta) \cdot \frac{\cosh(k \cdot (D+\eta)) \cdot \cosh(k \cdot z)}{\sinh^2(k \cdot (D+\eta))}$$

(18)

WAVE-EQUATION

The third order terms in the third order wave-equation are made of different contributions, here illustrated by examples. Third order terms in (10) and (13) get substitution of $\eta_1$, $Q_1$ with $R = k$ like from (11):

$$\left[(\frac{\partial Q}{\partial x})^2 - Q \cdot \frac{\partial^2 Q}{\partial x^2}\right] \cdot \frac{\partial \eta}{\partial x} \cdot$$

$$\left[- \frac{R \cdot \cosh(R \cdot (D+\eta)) \cdot [\cosh(2 \cdot R \cdot (D+\eta)) - \cosh(2 \cdot R \cdot z)]}{2 \cdot \sinh^3(R \cdot (D+\eta))}\right.$$

$$\left. + R \cdot \coth(R \cdot (D+\eta))\right] = (\frac{H}{2})^3 \cdot \omega^2 \cdot k \cdot$$

$$[\sin^2(\theta) + \cos^2(\theta)] \cdot \sin(\theta) \cdot k \cdot \coth(k \cdot (D+\eta)) \cdot$$

$$\left[- \frac{\cosh(2 \cdot k \cdot (D+\eta)) - \cosh(2 \cdot k \cdot z)}{2 \cdot \sinh^2(k(D+\eta))} + 1\right] \qquad (19)$$

Second order terms in (10) and (13) get substitution of $\eta_1$, $Q_1$ with $R = k$ combined with either $\eta_{2a}$, $Q_{2a}$ with $R = k$ or $\eta_{2b}$, $Q_{2b}$ with $R = 2 \cdot k$, like from (11) and (12):

$$\left[\frac{\partial Q}{\partial x} \cdot \frac{\partial^2 Q}{\partial x^2} - Q \cdot \frac{\partial^3 Q}{\partial x^3}\right] \cdot \frac{\cosh(2 \cdot R \cdot (D+\eta)) - \cosh(2 \cdot R \cdot z)}{4 \cdot \sinh^2(R \cdot (D+\eta))}$$

$$+ \left[\frac{\partial^2 Q_a}{\partial x^2} \cdot \frac{\partial Q_b}{\partial x} - Q_a \cdot \frac{\partial^3 Q_b}{\partial x^3}\right] \cdot \frac{R_a}{\sinh(R_a \cdot (D+\eta)) \cdot \sinh(R_b \cdot (D+\eta))}$$

$$\cdot \left[\frac{\cosh((R_b+R_a) \cdot (D+\eta)) - \cosh((R_b+R_a) \cdot z)}{2 \cdot (R_b+R_a)}\right.$$

$$\left. + \frac{\cosh((R_b - R_a) \cdot (D+\eta)) - \cosh((R_b - R_a) \cdot z)}{2 \cdot (R_b-R_a)}\right]$$

$$= \left[\frac{\partial Q_1}{\partial x} \cdot \frac{\partial^2 Q_{2a}}{\partial x^2} + \frac{\partial Q_{2a}}{\partial x} \cdot \frac{\partial^2 Q_1}{\partial x^2} - Q_1 \cdot \frac{\partial^3 Q_{2a}}{\partial x^3} - Q_{2a} \cdot \frac{\partial^3 Q_1}{\partial x^3}\right] \cdot$$

$$\frac{\cosh(2 \cdot k \cdot (D+\eta)) - \cosh(2 \cdot k \cdot z)}{4 \cdot \sinh^2(k \cdot (D+\eta))} + \left[\frac{\partial^2 Q_1}{\partial x^2} \cdot \frac{\partial Q_{2b}}{\partial x} - Q_1 \cdot \frac{\partial^3 Q_{2b}}{\partial x^3}\right] \cdot$$

$$\frac{k}{\sinh(k \cdot (D+\eta)) \cdot \sinh(2 \cdot k \cdot (D+\eta))} \cdot \left[\frac{\cosh(3 \cdot k \cdot (D+\eta)) - \cosh(3 \cdot k \cdot z)}{6 \cdot k}\right.$$

$$\left. + \frac{\cosh(k \cdot (D+\eta)) - \cosh(k \cdot z)}{2 \cdot k}\right] + \left[\frac{\partial^2 Q_{2b}}{\partial x^2} \cdot \frac{\partial Q_1}{\partial x} - Q_{2b} \cdot \frac{\partial^3 Q_1}{\partial x^3}\right] \cdot$$

$$\frac{2 \cdot k}{\sinh(2 \cdot k \cdot (D+\eta)) \cdot \sinh(k \cdot (D+\eta))} \cdot$$

$$\left[\frac{\cosh(3 \cdot k \cdot (D+\eta)) - \cosh(3 \cdot k \cdot z)}{6 \cdot k} - \frac{\cosh(k \cdot (D+\eta)) - \cosh(k \cdot z)}{2 \cdot k}\right]$$

$$= \left(\frac{H}{2}\right)^3 \cdot \frac{k^2}{2} \cdot \omega^2 \cdot \coth(k \cdot D) \cdot [6 \cdot \cos(\theta) \cdot \sin(2 \cdot \theta)$$

$$- 3 \cdot \sin(\theta) \cdot \cos(2 \cdot \theta)] \cdot \frac{\cosh(2 \cdot k \cdot (D+\eta)) - \cosh(2 \cdot k \cdot z)}{4 \cdot \sinh^2(k \cdot (D+\eta))}$$

$$+ \left(\frac{H}{2}\right)^3 \cdot \frac{3}{4} \cdot k^2 \cdot \omega^2 \cdot \frac{\coth(k \cdot D)}{\sinh^2(k \cdot D)} \cdot \{[- 2 \cdot \cos(\theta) \cdot \sin(2 \cdot \theta)$$

$$+ 8 \cdot \cos(\theta) \cdot \sin(2\cdot\theta)] \cdot \frac{k}{\sinh(k\cdot(D+\eta)) \cdot \sinh(2\cdot k\cdot(D+\eta))}$$

$$\cdot \left[\frac{\cosh(3\cdot k\cdot(D+\eta)) - \cosh(3\cdot k\cdot z)}{6 \cdot k} + \frac{\cosh(k\cdot(D+\eta)) - \cosh(k\cdot z)}{2 \cdot k}\right]$$

$$+ [- 4 \cdot \cos(2\cdot\theta) \cdot \sin(\theta) + \cos(2\cdot\theta) \cdot \sin(\theta)] \cdot$$

$$\frac{2 \cdot k}{\sinh(2\cdot k\cdot(D+\eta)) \cdot \sinh(k\cdot(D+\eta))} \cdot$$

$$\left[\frac{\cosh(3\cdot k\cdot(D+\eta)) - \cosh(3\cdot k\cdot z)}{6 \cdot k} - \frac{\cosh(k\cdot(D+\eta)) - \cosh(k\cdot z)}{2 \cdot k}\right]\Big\}$$

(20)

Third order terms are also got from the first and second order equations (15), (16) and (18). There are used:

$$\coth(R\cdot(D+\eta)) \simeq \coth(R\cdot D) - \frac{R \cdot \eta - R^2 \cdot \eta^2 \cdot \coth(R\cdot D)}{\sinh^2(R\cdot D)}$$

(21)

Until later when corrections are made the first and second order celerity, $C_1$ and $C_2$ will be used giving $g = \frac{\omega^2}{k} \cdot \coth(k\cdot D)$. (15) will then give a third order remainder on the left side of:

$$\left(\frac{H}{2}\right)^3 \cdot \frac{1}{2}\,\omega^2 k^2 \cdot \frac{\coth kD}{\sinh^2 kD} \cdot \sin\theta \cdot \cos 2\theta + \left(\frac{H}{2}\right)^3 \cdot \frac{3}{4}\,\omega^2 \cdot k^2 \cdot \frac{\coth kD}{\sinh^4 kD} \sin\theta \cos 2\theta$$

$$- \left(\frac{H}{2}\right)^3 \cdot \omega^2 \cdot k^2 \cdot \frac{\coth kD}{\sinh^2 kD} \sin\theta \cos^2\theta \qquad (22)$$

In the same way (16) gives:

$$\left(\frac{H}{2}\right)^3 \cdot \omega^2 \cdot k^2 \cdot \frac{\coth(k\cdot D)}{\sinh^2(k\cdot D)} \cdot \left[\frac{12}{\sinh^2(2\cdot k\cdot D)} - 8\right] \qquad (23)$$

$$\cdot \sin(\theta) \cdot \cos^2(\theta)$$

And (18):

$$\left(\frac{H}{2}\right)^3 \cdot 3 \cdot \omega^2 \cdot k^2 \cdot \frac{1}{\sinh^2(k \cdot D)} \cdot \frac{\cosh(k \cdot z)}{\sinh(k \cdot (D+\eta))} \cdot$$

$$\cos(\theta) \cdot \sin(2 \cdot \theta) \tag{24}$$

The wave-height, measured as the difference in the elevation of the crest and the trough, will not be the same for the complete third order wave and its first and second order components. The latter will then be written $H_1$ in the following to distinquish it from the real wave-height, $H$ (here: = third order wave-height), and the relation between $H$ and $H_1$ will be determined later. Then the complete wave-equation of third order will be written, first the (unknown) terms with $\eta_3$ and $Q_3$, then the terms with $\eta_{2a}$ and $Q_{2a}$, then with $\eta_{2b}$ and $Q_{2b}$, then with $\eta_1$, and $Q_1$, and finally the terms from (22), (23) and (24):

$$g \cdot \frac{\partial \eta_3}{\partial x} - \frac{\partial^3 Q_3}{\partial x^2 \cdot \partial t} \cdot \frac{\cosh(R \cdot (D+\eta)) - \cosh(R \cdot z)}{R \cdot \sinh(R \cdot (D+\eta))}$$

$$+ \frac{\partial Q_3}{\partial t} \cdot \frac{R \cdot \cosh(R \cdot z)}{\sinh(R \cdot (D+\eta))} + \left(\frac{H_1}{2}\right)^3 \cdot \frac{k^2}{2} \cdot \omega^2 \cdot \coth(k \cdot D)$$

$$\cdot \{- 2 \cdot \cos(\theta) \cdot \sin(2 \cdot \theta) - 4 \cdot \cos(2 \cdot \theta) \cdot \sin(\theta)$$

$$+ [20 \cdot \cos(\theta) \cdot \sin(2 \cdot \theta) + 25 \cdot \sin(\theta) \cdot \cos(2 \cdot \theta)]$$

$$\cdot \frac{\cosh(k \cdot (D+\eta)) \cdot [\cosh(k \cdot (D+\eta)) - \cosh(k \cdot z)]}{\sinh^2(k \cdot (D+\eta))}$$

$$+ [6 \cdot \cos(\theta) \cdot \sin(2\cdot\theta) - 3 \cdot \sin(\theta) \cdot \cos(2\cdot\theta)] \cdot$$

$$\frac{\cosh(2\cdot k\cdot(D+\eta)) - \cosh(2\cdot k\cdot z)}{4 \cdot \sinh^2(k\cdot(D+\eta))}$$

$$+ [2 \cdot \cos(\theta) \cdot \sin(2\cdot\theta) + \cos(2\cdot\theta) \cdot \sin(\theta)]$$

$$\left.\frac{\cosh(k\cdot(D+\eta)) \cdot \cosh(k\cdot z) + 1}{\sinh^2(k\cdot(D+\eta))}\right\}$$

$$+ \left(\frac{H_1}{2}\right)^3 \cdot \frac{3}{4} \cdot k^2 \cdot \omega^2 \cdot \frac{\coth(k\cdot D)}{\sinh^2(k\cdot D)} \cdot \left\{- 2 \cdot \cos(\theta) \cdot \sin(2\cdot\theta)\right.$$

$$- 4 \cdot \cos(2\cdot\theta) \cdot \sin(\theta)$$

$$+ [14 \cdot \cos(\theta) \cdot \sin(2\cdot\theta) + 12 \cdot \sin(\theta) \cdot \cos(2\cdot\theta)]$$

$$\cdot \frac{\cosh(k\cdot(D+\eta)) \cdot [\cosh(k\cdot(D+\eta)) - \cosh(k\cdot z)]}{\sinh^2(k\cdot(D+\eta))}$$

$$+ [13 \cdot \cos(2\cdot\theta) \cdot \sin(\theta) + 6 \cdot \sin(2\cdot\theta) \cdot \cos(\theta)] \cdot$$

$$\cdot \frac{\cosh(2\cdot k\cdot(D+\eta)) \cdot [\cosh(2\cdot k\cdot(D+\eta)) - \cosh(2\cdot k\cdot z)]}{\sinh^2(2\cdot k\cdot(D+\eta))}$$

$$+ 6 \cdot \cos(\theta) \cdot \sin(2\cdot\theta) \cdot \frac{k}{\sinh(k\cdot(D+\eta)) \cdot \sinh(2\cdot k\cdot(D+\eta))}$$

$$\cdot \left[\frac{\cosh(3\cdot k\cdot(D+\eta)) - \cosh(3\cdot k\cdot z)}{6 \cdot k} + \frac{\cosh(k\cdot(D+\eta)) - \cosh(k\cdot z)}{2 \cdot k}\right]$$

$$- 3 \cdot \cos(2\cdot\theta) \cdot \sin(\theta) \cdot \frac{2 \cdot k}{\sinh(k\cdot(D+\eta)) \cdot \sinh(2\cdot k\cdot(D+\eta))}$$

$$\cdot \left[\frac{\cosh(3\cdot k\cdot(D+\eta)) - \cosh(3\cdot k\cdot z)}{6 \cdot k} - \frac{\cosh(k\cdot(D+\eta)) - \cosh(k\cdot z)}{2 \cdot k}\right]$$

$$+ 2 \cdot \cos(\theta) \cdot \sin(2\cdot\theta) \cdot \frac{\cosh(k\cdot(D+\eta)) \cdot \cosh(k\cdot z)}{\sinh^2(k\cdot(D+\eta))}$$

$$+ \cos(2\cdot\theta) \cdot \sin(\theta) \cdot \frac{4 \cdot \cosh(2\cdot k\cdot(D+\eta)) \cdot \cosh(2\cdot k\cdot z)}{\sinh^2(2\cdot k\cdot(D+\eta))}$$

$$+ \cos(2\cdot\theta) \cdot \sin(\theta) \cdot \frac{\cosh(k\cdot z)}{\sinh(k\cdot(D+\eta))} \cdot \frac{2 \cdot \cosh(2\cdot k\cdot z)}{\sinh(2\cdot k\cdot(D+\eta))}$$

$$+ 2 \cdot \cos(\theta) \cdot \sin(2\cdot\theta) \cdot \frac{2 \cdot \cosh(2\cdot k\cdot z)}{\sinh(2\cdot k\cdot(D+\eta))} \cdot \frac{\cosh(k\cdot z)}{\sinh(k\cdot(D+\eta))}$$

$$- 2 \cdot \cos(\theta) \cdot \sin(2\cdot\theta) \cdot \frac{\sinh(k\cdot z)}{\sinh(k\cdot(D+\eta))} \cdot \frac{\sinh(2\cdot k\cdot z)}{\sinh(2\cdot k\cdot(D+\eta))}$$

$$- \cos(2\cdot\theta) \cdot \sin(\theta) \cdot \frac{4 \cdot \sinh(2\cdot k\cdot z)}{\sinh(2\cdot k\cdot(D+\eta))} \cdot \frac{\sinh(k\cdot z)}{\sinh(k\cdot(D+\eta))} \Big\}$$

$$+ \left(\frac{H_1}{2}\right)^3 \cdot \omega^2 \cdot k \cdot \Big\{[- 2 \cdot \sin^2(\theta) + \cos^2(\theta)] \cdot \sin(\theta) \cdot$$

$$\Big[- \frac{k \cdot [\cosh^2(k\cdot(D+\eta)) + 1] \cdot [\cosh(k\cdot(D+\eta)) - \cosh(k\cdot z)]}{\sinh^3(k\cdot(D+\eta))}$$

$$+ k \cdot \coth(k\cdot(D+\eta))\Big] + [\sin^2(\theta) - 2 \cdot \cos^2(\theta)] \cdot \sin(\theta) \cdot$$

$$\frac{k \cdot [\cosh(k\cdot(D+\eta)) - \cos(k\cdot z)]}{\sinh^3(k\cdot(D+\eta))} \cdot [\cosh^2(k\cdot(D+\eta)) + 1]$$

$$+ \sin(\theta) \cdot k \cdot \coth(k\cdot(D+\eta)) \cdot$$

$$\Big[- \frac{\cosh(2\cdot k\cdot(D+\eta)) - \cosh(2\cdot k\cdot z)}{2 \cdot \sinh^2(k\cdot(D+\eta))} + 1\Big]$$

$$- 3 \cdot \cos^2(\theta) \cdot \sin(\theta) \cdot k \cdot \coth(k\cdot(D+\eta)) \cdot$$

$$\frac{\cosh(2\cdot k\cdot(D+\eta)) - \cosh(2\cdot k\cdot z)}{4 \cdot \sinh^2(k\cdot(D+\eta))}$$

$$- \cos^2(\theta) \cdot \sin(\theta) \cdot \frac{k \cdot \coth(k\cdot(D+\eta))}{\sinh^2(k\cdot(D+\eta))}\Big\}$$

$$+ \left(\frac{H_1}{2}\right)^3 \cdot \omega^2 \cdot k^2 \cdot \Big\{- \sin(\theta) \cdot \cos^2(\theta) \cdot \frac{\coth(k\cdot D))}{\sinh^2(k\cdot D)}$$

$$+ \frac{1}{2}\cdot\sin\theta\cdot\cos 2\theta\cdot\frac{\coth kD}{\sinh^2 kD} + \frac{3}{4}\cdot\sin\theta\cdot\cos 2\theta\cdot\frac{\coth kD}{\sinh^4 kD}$$

$$+ \sin(\theta) \cdot \cos^2(\theta) \cdot \frac{\coth(k \cdot D)}{\sinh^2(k \cdot D)} \cdot \left[\frac{12}{\sinh^2(2 \cdot k \cdot D)} - 8\right]$$

$$+ 3 \cdot \cos(\theta) \cdot \sin(2 \cdot \theta) \cdot \frac{1}{\sinh^2(k \cdot D)} \cdot \frac{\cosh(k \cdot z)}{\sinh(k \cdot (D+\eta))}\Big\}$$

$$= 0 \qquad (25)$$

## SOLUTIONS

In (25) the harmonic functions can be substituted by $\sin(3 \cdot \theta)$ and $\sin(\theta)$. It is seen on the equation that it can not be solved for terms of $\sin(\theta) \cdot \cosh(k \cdot z)$ and $\sin(3 \cdot \theta) \cdot \cosh(3 \cdot k \cdot z)$, and as a check on the calculations it is further seen that the result of those terms is 0. (25) is divided into 5 equations: one of $\sin(3 \cdot \theta) \cdot \cosh(k \cdot z)$, one of $\sin(3 \cdot \theta) \cdot \cosh(2 \cdot k \cdot z)$, one of $\sin(\theta) \cdot \cosh(2 \cdot k \cdot z)$, and two z-independent equations. The three z-dependent equations give the solutions $\eta_{3a}$, $\eta_{3b}$ and $\eta_{3c}$. First $\eta_{3a}$ with $R = k$:

$$\eta_{3a} = \left(\frac{H_1}{2}\right)^3 \cdot \frac{k^2}{8} \cdot [3 \cdot \coth^4(k \cdot D) - \coth^2(k \cdot D) + 1]$$

$$\cdot \cos(3 \cdot \omega \cdot t - 3 \cdot k \cdot x) \qquad (26)$$

Then $\eta_{3b}$ with $R = 2 \cdot k$:

$$\eta_{3b} = \left(\frac{H_1}{2}\right)^3 \cdot \frac{3}{8} \cdot k^2 \cdot [\coth^4(k \cdot D) - 1] \cdot$$

$$\cos(3 \cdot \omega \cdot t - 3 \cdot k \cdot x) \qquad (27)$$

Then the solution with a surface-profile like the one of first order, $\eta_{3c}$ with $R = 2 \cdot k$:

$$\eta_{3c} = \left(\frac{H_1}{2}\right)^3 \cdot k^2 \cdot \left[\frac{3}{8} \cdot \coth^4(k \cdot D) + \frac{1}{6} \cdot \coth^2(k \cdot D) - \frac{3}{8}\right] \cdot \cos(\omega \cdot t - k \cdot x) \qquad (28)$$

The two z-independent equations are: one with $\sin(\theta)$ and one with $\sin(3 \cdot \theta)$. Besides of z-independent terms from (25) they consist of terms from $\eta_{3a}$, $\eta_{3b}$, and $\eta_{3c}$ substituted into:

$$g \cdot \frac{\partial \eta_3}{\partial x} - \frac{\partial^3 Q_3}{\partial x^2 \cdot \partial t} \cdot \frac{1}{R} \cosh(R \cdot (D+\eta))$$

$g$ is still substituted by $g = \frac{\omega^2}{k} \cdot \coth(k \cdot D)$. The solution to the equation of $\sin(3 \cdot \theta)$ must have $R = 3 \cdot k$ because of the z-dependent equations:

$$\eta_{3d} = \left(\frac{H_1}{2}\right)^3 \cdot \frac{1}{64} \cdot k^2 \cdot [18 \cdot \coth^6(k \cdot D) - 33 \cdot \coth^4(k \cdot D) + 8 \cdot \coth^2(k \cdot D) + 7] \cdot \cos(3 \cdot \omega \cdot t - 3 \cdot k \cdot x) \qquad (29)$$

# CELERITY AND WAVE-HEIGHT

The z-independent equation with $\sin(\theta)$ will reduce to:

$$- g \cdot \frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial x \cdot \partial t^2} \cdot \frac{1}{R}\coth(R\cdot D)$$

$$= (\frac{H_1}{2})^3 \cdot k^2 \cdot \omega^2 \cdot \coth(k\cdot D) \cdot \sin(\theta) \cdot$$

$$\left[1\frac{1}{8} \cdot \coth^4(k\cdot D) - 2\frac{1}{4} \cdot \coth^2(k\cdot D) + 1\frac{11}{24}\right] \qquad (30)$$

Because of the z-dependent equations $R = k$. But then no solution of third order can satisfy (30) with the usual $g$. So $g$ must be different and then primarily for the first order term of $\eta$. $g$ will then be written:

$$g = \alpha \cdot \frac{\omega^2}{k} \cdot \coth(k\cdot D) \qquad (31)$$

whereby (30) with $\eta = \eta_1 = \frac{H}{2} \cdot \cos(\theta)$ give for $\alpha$:

$$\alpha = 1 - (\frac{H_1}{2})^2 \cdot k^2 \cdot \left[1\frac{1}{8} \cdot \coth^4(k\cdot D) - 2\frac{1}{4} \cdot \coth^2(k\cdot D) + 1\frac{11}{24}\right] \qquad (32)$$

where there can be used $H_1 = H$. (31) gives the celerity $C_3$ for the wave of third order:

$$C_3^2 = \frac{\omega^2}{k^2} = \frac{1}{\alpha} \cdot \frac{g}{k} \cdot \tanh(k\cdot D) \qquad (33)$$

Deep-water waves ($D \to \infty$) give:

$$\alpha = 1 - \frac{\pi^2}{3} \cdot (\frac{H_1}{L})^2 \qquad (34)$$

from which it is seen that for deep-water waves of even 10% steepness the celerity is increased less than 2% by using the third order theory instead of first or second order.

Like shown in chapter VII for the wave of second order the particle velocities and pressure can now be calculated by (2), (3), and (7) using $Q = C \cdot \eta$ with the different R-values and $\eta$- values and using:

$$\eta = \eta_1 + \eta_{2a} + \eta_{2b} + \eta_{3a} + \eta_{3b} + \eta_{3c} + \eta_{3d} \tag{35}$$

The wave-height, H, is then written as a function of $H_1$ by (26), (27), (28) and (29):

$$\frac{H}{2} = \frac{H_1}{2} + \left(\frac{H_1}{2}\right)^3 \cdot k^2 \cdot \left[\frac{9}{32} \cdot \coth^6(k \cdot D) + \frac{39}{64} \cdot \coth^4(k \cdot D) + \frac{1}{6} \cdot \coth^2(k \cdot D) - \frac{33}{64}\right] \tag{36}$$

which for $D \to \infty$ is reduced to:

$$\frac{H}{2} = \frac{H_1}{2} + \left(\frac{H_1}{2}\right)^3 \cdot k^2 \cdot \frac{13}{24} \tag{37}$$

from which it is seen that $H_1$ is about 5% less than H for a deep-water wave of 10% steepness.