CHAPTER I

PRACTICAL CONSIDERATIONS ON REGULAR WAVES

INTRODUCTION

Regular waves have been known for many years in hydrodynamics, and many papers have been written on this subject through the years. But still it is felt that a few things in the wave expressions in common use are not completely satisfactory.

In this chapter we will make some considerations on regular wave behaviour from an engineering point of view. We will consider some of the most used expressions and see how easy they can be changed to give better results in specific cases.

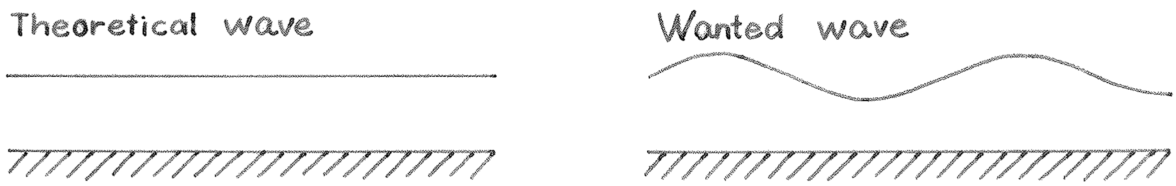

Fig. 1. The engineer has primarily an interest in the big waves. So if it is necessary in the theory to anticipate, that the waves are infinite small, the results found in this way should also be reasonable for the big waves.

PRACTICAL CONSIDERATIONS

Let us first of all consider the wave height. An engineer working with practical problems in coastal hydraulics cannot always be content with infinitely small waves. When designing coastal structures, the design wave will be rather high, so that a wave height H, bigger than half the mean water depth D is not at all unrealistical.

This may seem to make it very difficult to make a wave theory, because we will have to assume a small wave height in the development of our theories. But we can get around this problem. When we develope a wave theory under the assumption of small waves we must try to adjust the final expressions so that they also yield reasonable results for the big waves. This can be done arbitrarily within certain limits at the end, or the whole theory can from the beginning be made with this purpose in mind.

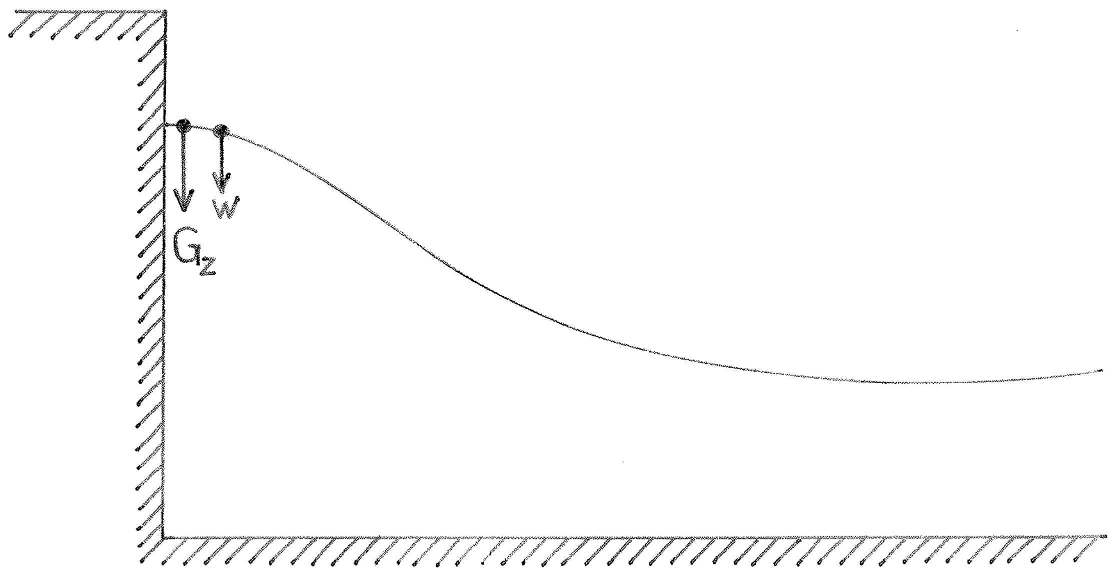

Fig. 2. For the engineer it is rather easy to observe the oscillations of the water surface at the vertical wall. Because of that, he would like the final expressions for the vertical velocity and the vertical acceleration to agree with the expression found directly from the oscillation of the surface. This is also of big importance in deciding the wave pressure on the vertical wall.

Let us as an example consider the vertical particle velocity at the vertical wall. The Airy theory gives for the surface profile of the standing wave

$$\eta = \frac{H}{2} \cos\omega t \cos kx \tag{1}$$

where $\omega = 2\pi/T$ and $k = 2\pi/L$ with T and L being the wave period and the wave length. The horizontal coordinate x has got $x = o$ at the wall.

The Airy expression for the vertical particle velocity is

$$w = -\frac{\pi H}{T} \frac{\sinh kz}{\sinh kD} \sin\omega t \cos kx \tag{2}$$

where z is the vertical coordinate with $z = o$ at the bottom. The water close to the wall will then move up and down (without any horizontal velocity). Two particles are of special practical interest, the particle at the bottom and the particle at the surface. At the bottom, $z = o$, eq. 2 gives $w = o$, as wanted. At the surface $z = D+\eta$, eq. 2 gives for $x = o$ (the wall)

$$w_{s1} = -\frac{\pi H}{T} \frac{\sinh k(D+\eta)}{\sinh kD} \sin\omega t \tag{3}$$

But to find the vertical velocity of the surface particle it would be just as natural to differentiate eq. 1 and get

$$w_{s2} = \frac{\partial\eta}{\partial t} = -\frac{H}{2}\omega \sin\omega t = -\frac{\pi H}{T} \sin\omega t \tag{4}$$

comparing eqs. 3 and 4 we get

$$\frac{w_{s1}}{w_{s2}} = \frac{\sinh k(D+\eta)}{\sinh kD} \tag{5}$$

For infinite small waves, $\eta/D \cong o$, this expression gives the wanted result $w_{s1}/w_{s2} \cong 1$. But for more realistic waves eq. 5 may give $w_{s1}/w_{s2} > 1.3$. If we for the surface particle considered the vertical acceleration, G_z, in the same way we would find the same result.

A difference of 30 % in the numerical results from calculating w and G_z after two different expressions is not always so easy to accept. And it does not need to be accepted. The whole problem can be solved by changing eq. 2 to

$$w = - \frac{\pi H}{T} \frac{\sinh kz}{\sinh k(D + \eta)} \sin \omega t \cos kx \qquad (6)$$

which can be written

$$w = \frac{\partial \eta}{\partial t} \frac{\sinh kz}{\sinh k(D + \eta)} \qquad (7)$$

This expression is the natural result of the first order sinusoidal theory of chapter IV. Within first order approximation eqs. 2 and 7 are identical. Actually it would be permitted in wave hydrodynamics simply arbitrarily to change eq. 2 to eq. 7, e.g. with the purpose of fulfilling the surface condition considered above. The difference between eqs. 2 and 7 is that eq. 7 uses the actual water depth $D + \eta$ instead of just the mean water depth D as used by eq. 2.

The acceleration of the surface is

$$G_{zs} = \frac{\partial^2 \eta}{\partial t^2} \qquad (8)$$

so for the water at the vertical wall it will be reasonable to propose the vertical acceleration

$$G_z = \frac{\partial^2 \eta}{\partial t^2} \frac{\sinh kz}{\sinh k(D + \eta)} \qquad (9)$$

This is done after the same considerations that led to eq. 7. This expression is the same as will be found from the Airy theory (differentation of w in eq. 2) within first order approximation. The reason to use eq. 9 instead of the Airy expression is again that it gives the right result at the surface.

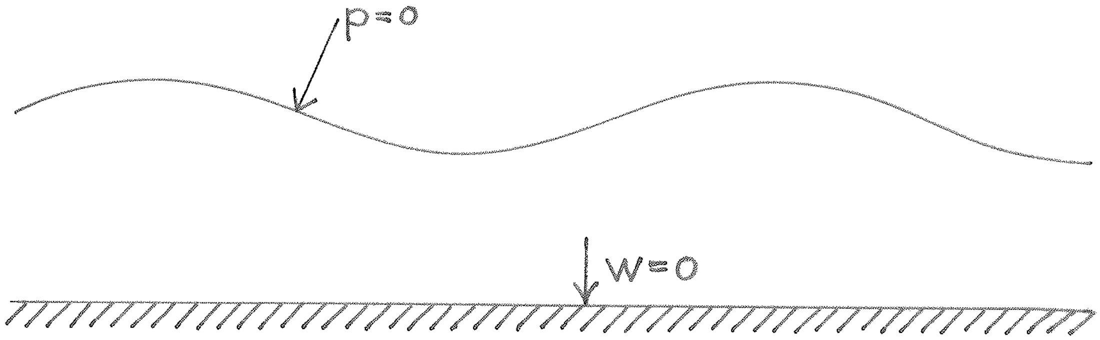

Fig. 3. The engineer wants spontaneously that the final expression for the vertical particle velocity gives $w = o$ at the horizontal water tight bottom. This demand is traditionally fulfilled exactly. In the same way the engineer wants the water pressure at the surface to be $p = o$. By numerical use of traditional formulas this demand is usually not fulfilled neither for the progressive nor the standing wave.

Another example of big interest for the practical engineer is the pressure caused by the waves. If the fluid pressure (pressure in excess of the atmospheric pressure) is called p the wave pressure is

$$p^+ = p - \gamma(D - z) \tag{10}$$

where γ is the unit weight. The Airy expression is

$$\frac{p^+}{\gamma} = \frac{H}{2}\frac{\cosh kz}{\cosh kD}\cos kx \cos\omega t \tag{11}$$

or for both the progressive and the standing wave

$$\frac{p^+}{\gamma} = \eta\frac{\cosh kz}{\cosh kD} \tag{12}$$

This expression is short and simple to use. But it is not without problems.

If we consider a standing wave with the crest at the wall the pressure at the mean water level, $z = D$ will be

$$\frac{p^+}{\gamma} = \frac{p}{\gamma} = \eta \tag{13}$$

which is the same as the hydrostatic pressure from the crest above. This means that the pressure reducing effect from the negative (vertical) acceleration of the water above is neglected. For bigger waves the vertical acceleration is important, and the highest possible standing wave can actually be determined by the criterion that the negative acceleration cannot exceed the acceleration of gravity, g. If the whole crest had this big acceleration the pressure at the mean water level would simply be $p = o$. So for the engineer that works with the high design waves, eq. 13 is not so economical. Instead the acceleration of eqs. 8 and 9 together with the equation of momentum will give a better expression

$$\frac{p}{\gamma} = D + \eta - z + \frac{G_{zs}}{g} \frac{1}{k} \frac{\cosh k(D+\eta) - \cosh kz}{\sinh k(D+\eta)} \qquad (14)$$

(It is shown in detail how to find this expression in chapter IV, where it also will be shown that eqs. 12 and 14 are identical within first order approximations).

Eq. 14 solves other problems, too. Eq. 12 cannot be used above the mean water level, for $z > D$. This part of the wave is important, e.g. in determining the overturning moment on a vertical face breakwater. Eq. 14 can as well be used for $z > D$. At the surface of the water, $z = D + \eta$, eqs. 12 and 1o will not give $p = o$ exactly. This needs then to be when the wave has trough, for $\eta < o$. Eq. 14 is seen to give the wanted $p = o$ at the surface for any η.

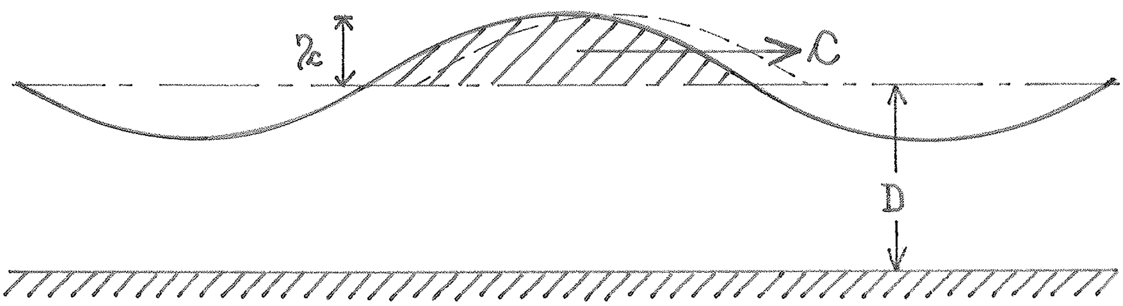

Fig. 4. When observing progressive waves of permanent form it looks as if the shown part of the crest slides on 'frozen' water. The trough is considered as a sliding negative crest. In this way the water discharge through a vertical is determined.

Let us now consider the water discharge q through a vertical for a progressive wave of permanent form. The spontaneous impression when observing progressive waves is that the waves (the water of the waves) moves with the velocity c, the wave celerity. It seems as if the crest as a solid body slides on the calm water below, where the water below the mean water level can be considered as a solid stationary body. The trough is considered in the same way as a negative crest sliding on the water below. Such considerations do not represent reality with concern to e.g. momentum, but it is an obvious and very practical way to determine the waterdischarge created by the wave.

The vertical through the crest will then have a q of

$$q = c\,\eta_c \tag{15}$$

where η_c is the crest height. The water does of course not only pass through the upper η_c with the big constant velocity c. The water passes through the whole vertical $D + \eta_c$ with the much smaller variable velocity, u, so that q will be

$$q = \int_0^{D+\eta_c} u\,dz \tag{16}$$

At any other place the discharge is found in the same way as in eq. 15 to

$$q = c\,\eta \tag{17}$$

which as in eq. 16 will give

$$q = c\,\eta = \int_0^{D+\eta} u\,dz \tag{18}$$

In this simple and practical manner we have a very basic and natural equation for the horizontal velocity of a pure wave.

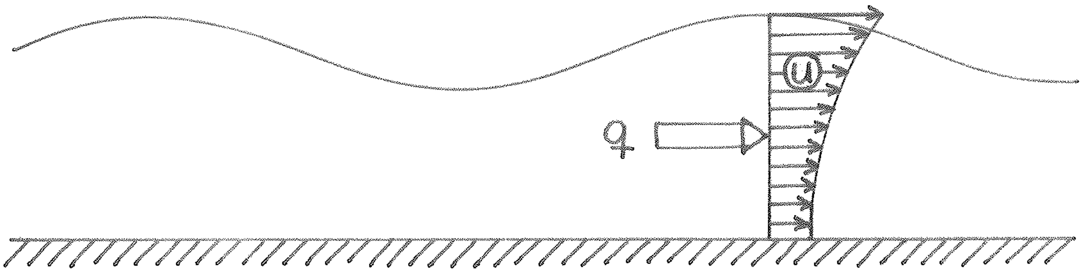

Fig. 5. When the engineer wants to find the water discharge in a progressive wave it is simple for him to express it as $q = c\eta$ (see eq. 17). But it is just as natural to integrate the horizontal velocity profile, u.
The two expressions is then wanted to give the same numerical results, but this is traditionally not the situation.

But eq. 18 is not fulfilled exactly in the traditional wave theories. The first order progressive wave is

$$\eta = \frac{H}{2} \cos k(x - ct) \tag{19}$$

The Airy expression for u is

$$u = \frac{\pi H}{T} \frac{\cosh kz}{\sinh kD} \cos k(x - ct)$$

$$= c\,\eta\, k \frac{\cosh kz}{\sinh kD} \tag{2o}$$

This gives by integration

$$q = \int_0^{D+\eta} u\,dz = c\,\eta \frac{\sinh k(D + \eta)}{\sinh kD} \tag{21}$$

If the engineer for realistic waves compares eq. 21 with eq. 17 he may again find that eq. 21 with numerical examples yield values that are more than 3o % too big. In traditional cnoidal wave theories the difference may be found even much bigger.

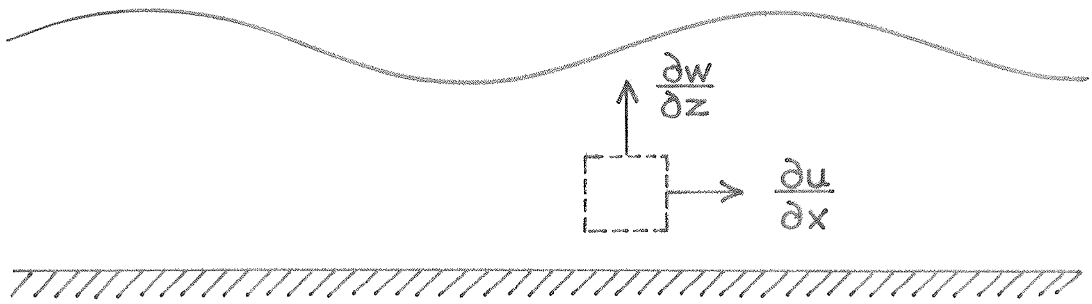

Fig. 6. In every point of the fluid the equation of continuity must be fulfilled. But as the engineer seldom needs both velocity gradients he will not find a spontaneous reason to verify the equation of continuity. So it can better be accepted that this equation in the final expressions is fulfilled only with hydrodynamic approximations, than there are approximations in the boundary conditions.

The next equation we will consider is the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{22}$$

It is considered a very important equation in hydrodynamics. It is simple to find. The flow through an infinitesimal box is considered. The equation is also simple to use. But still it is felt that for the engineer the exact fulfilment of this equation is not quite as important as the boundary equations mentioned above. This is because the engineer will rather seldom find it necessay to use both $\partial u/\partial x$ and $\partial w/\partial z$ at the same time. So he will usually in practice not be faced with numerical deviations in the situation where the equation of continuity is only fulfilled approximately.

So when it is necessary to make approximations in the wave theory it is better to make the approximations here than in the boundary conditions considered above.

For the wave theories of this thesis it is though not necessary in the final formulas to have the equation of continuity fulfilled only within the given order of approximation.

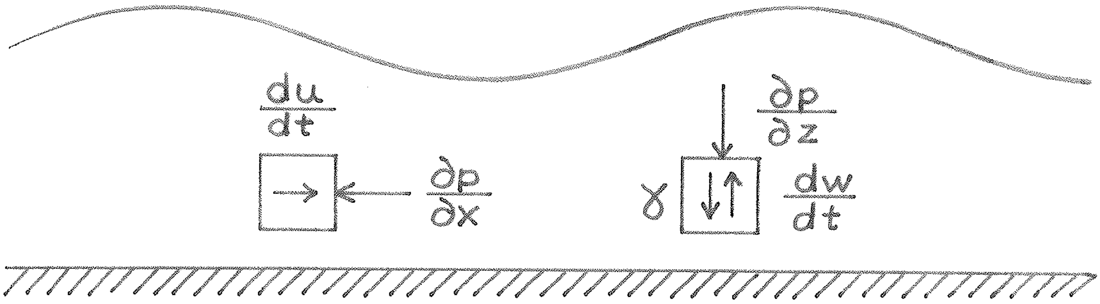

Fig. 7. In every point of the fluid the equations of momentum in both horizontal and vertical direction must be fulfilled. But of the same reason as for the equation of continuity the fulfilment can be accepted to be only hydrodynamically approximate.

In the equations of momentum it will usually be necessary to make approximations of first or second order magnitude, etc. giving a so-called first order or second order wave.

The horizontal equation of momentum is

$$\rho \frac{du}{dt} = - \frac{\partial p}{\partial x} \tag{23}$$

and the vertical equation of momentum is

$$\rho \frac{dw}{dt} = - \gamma - \frac{\partial p}{\partial z} \tag{24}$$

It is felt that usually there will not be spontaneous reason for the engineer to reach a deviation in numerical results through those two equations. Otherwise the next chapters will indicate how the final expression for p should be given to have eq. 24 fulfilled exactly. But in the theories of this thesis one equation has got to be fulfilled only approximately. Otherwise we had the ideal exact wave theory.

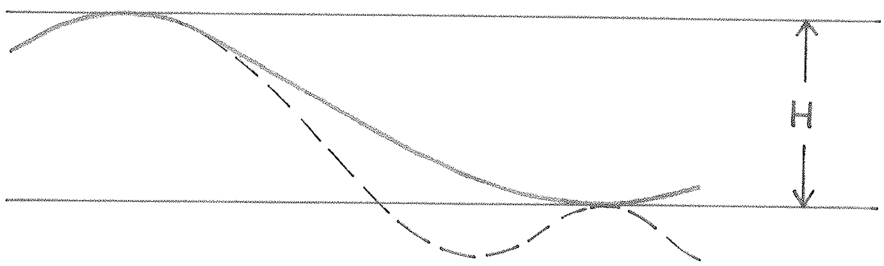

Fig. 8. For a wave with the wave height, H, we can think of a situation as shown with a given crest level, so that H lower we have the trough level. The wave profile must then be kept within these two levels, and further for a regular wave the engineer wants the profile to drop smooth and gradual from the crest level to the trough level. For this purpose the cosine function is good as used in the first order sinusoidal wave, while the superposition of higher harmonic functions (the Stokes' theory) better can be substituted by the new cnoidal theory of this thesis.

We have now been through some practical considerations of the basic hydrodynamic conditions in wave theories, the conditions we will use in the next chapters. A condition of most importance in the classical wave theories has not been considered here so far : the rotation condition. The Airy wave theory demands first of all the motion to be irrotaional. But for the waves of this thesis the rotation does not need to be considered to develope wave theories. The rotation can be considered at the end.

CONCLUSION

The waves we will develope in this thesis will not be exact. They cannot fulfil all the hydrodynamic conditions exactly at the same time. So somewhere there must be approximations, just like there are approximations in the traditional wave theories. But this does not mean that the same order of approximations should be accepted anywhere. As indicated with the considerations above it would be of practical value to avoid approximations in certain boundary conditions, at least as long as this can be done without making further approximations elsewhere. So when we in the following have a second order wave it means that we have made approximations somewhere of second order magnitude, but the approximations are never bigger than in a Stokes' second order wave.