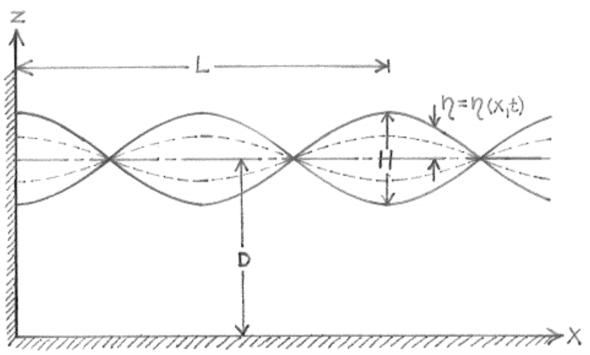
Wave pressure in a standing wave

When moderate regular ocean waves travel towards a vertical face breakwater we observe that the waves become standing waves: the water at the vertical wall oscillates regular up to a crest and down to a trough.



The height from the trough to the crest is the wave height H. The time from one crest at the wall to the next crest is the wave period T. At a distance from the wall we call the wave length L the water will oscillate with a crest at the same time as at the wall, while at the distance L/2 from the wall the water will oscillate opposite also with the wave height H. (L depends on the wave period T and the mean water depth, as decided by the wave theory).

Water pressure example

What will the water pressure on the wall be when there is a wave crest and when there is a wave trough? When there are no waves the water pressure 1 meter below the surface is = 1 meter water pressure (= 10 kN/m^2). There is hydrostatic water pressure from the surface and down to the bottom, so that with 10 m water depth the pressure at the bottom is 10 m water pressure.
If the water at the wall has risen 5 m because of a high tide lasting for hours then this calm water will have a water pressure at 1 m depth of 1 m (= 1000 kg × acceleration of gravity 9,81 $m/sek^2 \simeq 10\ kN/m^2$ pressure), and at the bottom 15 m water pressure.
If the water instead because of T = 6 seconds storm waves from the ocean has risen to a 5 m crest in a short second with a downward turning acceleration of 5 m/sec^2 reducing the acceleration of gravity then the water pressure at 1 m depth is only ½ m = 5 kN/m^2. This follows of Newton's 2' law (momentum). And this is shown in our wave theory here.
With this moderate wave giving a crest at the wall of 5 m above the D = 10 m mean water depth then the pressure at the bottom will not be 10 + 5 = 15 m hydrostatic water pressure. The water by the wall is turning from upward movement to downward movement. So the water has a downward vertical acceleration, giving by Newton's 2' law that the pressure at the bottom will be less than 15 m water pressure. This less bottom pressure is also a result of the classical wave theories by Airy as well as Stokes from the 1800-s.

Considering the water pressure 1 m below the surface of the wave crest we found that it would be less than 1 m, and for the waves of design interest: the high waves, the pressure will be even less than ½ m, and this cannot be seen in the classical wave theories. So we want a better expression for the wave pressure, a formula that fulfills the surface conditions for pressure and acceleration and Newton's 2' law.

A different wave theory

When designing a vertical face breakwater we want to know what wave pressure it will get. And more: we would like to know the wave pressure and water velocities everywhere in the water, from the surface to the bottom and all over the wave length. A different wave theory based on necessary practical approximations in its theoretical development will be given with formulas here.

Approximations

For any flow of water we have the equation of continuity (conservation of mass), and we have the dynamic equation Newton's 2' law, to be used horizontally and vertically. But to get our wave theory we have to make some assumptions and some approximations:
At the sea bottom the water will move somewhat back and forth, with a little friction, so we may see some bottom sand moving a little. In our theory we neglect that friction, and also friction at the wall, and internal friction. We have an ideal fluid. (When our wave theory gives the horizontal water velocity at the bottom we have the possibility to maybe make moderations as practical engineers).

In our simple so called 1' order wave theory we consider the wave height H as small compared to the wave length L. So terms of higher order in the wave steepness H/L can be neglected.
When a 1' order theory is developed we can develop a 2' order theory by using the formulas of the 1' order theory in the 2' order terms.

At the wall the water moves vertically up and down, which according to the 1' order theory follows the sine (sinus) or cosine as a function of time t. By this we can calculate the vertical velocity and acceleration of the surface water. At the bottom the vertical velocity and acceleration is = 0. Then we can propose and try to prove in our wave theory if we can use that the acceleration from the surface and decreasing down to the bottom can be distributed as hyperbolic sine (sinh), that is all the way from the surface of the wave crest with a possible big negative acceleration and down to the bottom (and not just from the mean water level MWL and down). This will according to Newton's 2' law give that the wave pressure on the wall will have a cosh distribution all the way from the crest surface, giving e.g. that the wave pressure at the mean water level is less than hydrostatic pressure.

It is then to be proved that using this sinh vertical distribution and the equation of continuity and Newton's 2' law of momentum we will get the wave equation for a regular standing wave.

The mathematical development of the wave theory is given on: http://lavigne.dk/waves/Ch5.pdf
This is chapter V in my book on more new wave theories: "Regular Waves, 1977":
http://lavigne.dk/waves/wavesd.htm, or use my home page: www.mejlhede.dk

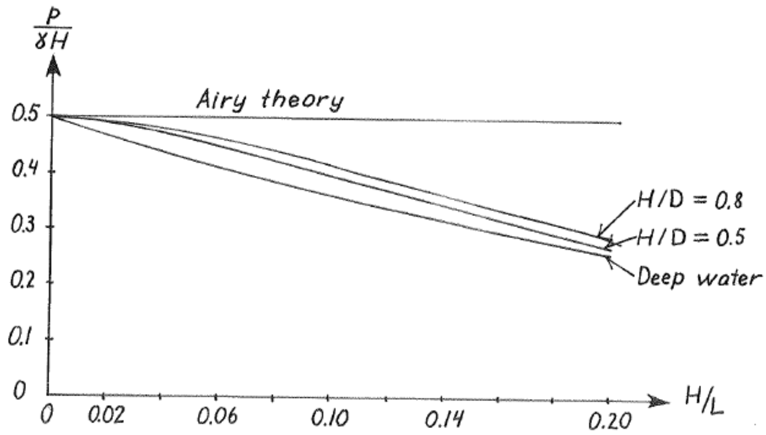


Figure 2: Wave pressure on a vertical wall at mean water level (MWL) below the wave crest of a regular wave, according to the 1' order theory. With a wave height of e.g. H = 1 meter the crest height is = 0.5 meter, so this gives a wave pressure of 0.5 m water (= 5 kN/m^2) according to the Airy theory.
Airy's classical theory does not include the pressure reducing effect of the "turning" acceleration of the above crest, which is included in my wave theory, and this is better according to experiments. Formula on the next page, illustrated in this figure for different wave steepness H/L and different mean depth D.

Formulas for the simple 1' order standing wave

In a coordinate system x,z with 0,0 at the foot point of the wall the equation for the surface profile measured from the mean water level MWL at z = D for the regular 1' order standing wave is:

$\eta = H/2 \cos(\omega t) \cos(kx)$

For a water particle with the coordinates x,z and time t we get the following formulas for water pressure p, and vertical velocity w, and vertical acceleration G_z , to be used anywhere in the water:

$p/\gamma = y - z + (G_s/g) \times (\cosh(Ry) - \cosh(Rz)) / (R \sinh(Ry))$ for water pressure anywhere, and for wave pressure p^+/γ above MWL

From this p/γ we get the wave pressure p^+/γ by subtracting the hydrostatic pressure from MWL: $p/\gamma = D - z$ (This hydrostatic pressure is the same as the water pressure on the calm harbor side of the breakwater).

Above the trough from surface to MWL we have negative hydrostatic pressure: $p^+/\gamma = p/\gamma = z - D$

Below the surface of trough and below MWL ($z \lesssim D$) for crest we get the wave pressure:

$p^+/\gamma = \eta + (G_s/g) \times (\cosh(Ry) - \cosh(Rz)) / (R \sinh(Ry))$

$w = \partial\eta/\partial t \times \sinh(Rz) / \sinh(Ry)$

$G_z = \partial^2\eta/\partial t^2 \times \sinh(Rz) / \sinh(Ry)$; at the surface z= y: $G_s = \partial^2\eta/\partial t^2$

$(L/T)^2 = g/k \times \tanh (kD)$

$q = H/2 \times L/T \times \sin(\omega t) \sin(kx)$

$u = q \times R \times \cosh(Rz) / \sinh(Ry)$

$y = D + \eta$, where D = mean water depth, so y = actual water depth.

For the regular 1' order wave we have: $\partial^2\eta/\partial t^2 = - \omega^2 \times \eta$, $R = k = 2\pi/L$, $\omega = 2\pi/T$
γ = weight of water 10 kN/m^3, g = acceleration of gravity = 9,81 $m/sec^2 \approx 10$ m/sec^2

Formulas for p^+, w, G, and u are of 1' order approximations, and in developing the wave theory the distribution of one of them is estimated and assumed, like sinh for the vertical acceleration, and then the theory gives the other formulas. (Or we can instead assume u to be cosh distributed).

For the surface z = y (= D + η) we have:
pressure p = 0, vertical velocity $w_s = \partial\eta/\partial t$, vertical acceleration $G_{z=s} = G_s = \partial^2\eta/\partial t^2$
Wave pressure at the surface: $p^+/\gamma = 0$ above MWL, and $p^+/\gamma = \eta$ (negative) at wave trough surface.

We get the simple traditional classic Airy formula for wave pressure by substituting y = D in the above formula, as η in a 1' order theory is considered small and neglected:
$p^+/\gamma = \eta \times \cosh(kz) / \cosh(kD)$, but for z > D (above MWL) hydrostatic pressure is used.
This gives a bigger wave pressure than in my experiments and by my formula.
At the surface of the wave trough the Airy formula does not give the water pressure p = 0.

1’ order wave pressure by practically correcting with 2’ order terms

At the wall: cos(kx) = 1.

A 1’ order wave has crest at the wall with η_c = +H/2 and trough with η_t = -H/2.

In 1968 I made some experiments at The Technical University in Copenhagen measuring wave pressure on a vertical wall by measuring the total horizontal sliding force and the overturning moment for e.g. designing the stability foundation of a breakwater. From figure 3 and 4 it is seen that the simple 1’ order wave pressure formula reasonable agrees with experiments, but for low steepness H/L the formula for wave sucking (negative pressure) gives a too big result.

The 2’ order sinusoidal formula for vertical acceleration is complicated and not suitable for the wave trough with its extra 2’ order wave crest on top of the trough. But it shows that the vertical acceleration follows sinh(kz), except for a minor part sinh(2kz), (shown in chapter VII in the book http://lavigne.dk/waves/wavesd.htm).

But as the sinusoidal 2’ order theory shows for particular less steep waves, and likewise the cnoidal wave theory, then the crest η_c is higher than H/2 and the trough η_t less deep than H/2 because of 2’ order terms, terms that in a 1’ order theory were regarded small and negligible. The less deep trough therefore gives a less big negative force. But does the higher crest then not give a too big pressure (compared to experiments)? No, the “narrow” short high crest has compared to the 1’ order wave at the same time a bigger vertical acceleration that moderates the pressure. (For the cnoidal wave theory see chapter VIII and IX in the book).

Let us as practical engineers try to use that to modify our 1’ order wave theory, because we prefer to work with the 1’ order simple formulas, now in a modified 1’ order theory:
So we increase the crest and decrease the trough with 2’ order values:

On water D = 10 m we consider a cnoidal wave, T = 10 sec, H = 6 m. It is a wave of second order.
It has crest η_c = +4.3 m and trough η_t = -1.7 m, so H/2 = 3 m is changed with a 2’ order term of 1.3 m. This is an important practical change, but in the wave theory this is only a (negligible) second order term that makes no difference in the theoretical development of a correct 1’ order wave, so its usual formulas can still be used.

The changed η_c and η_t are now used in the 1’ order formula for pressure. But compared to experiments the formula now gives too big a pressure from the high wave crest η_c. This is because the high narrow crest also has a bigger pressure modifying acceleration to be considered. So we change the acceleration term with R = 2π/L by using a shorter L, using L_c where $L_c/4$ = is the distance from wave crest to where η = 0. And for the wave trough we use $L_t/4 = L/2 - L_c/4$. These changes of L is also just changes of 2’ order in 1’ order formulas. With these practical changes of 2’ order included we still have a correct 1’ order wave theory.

This manner of making the 1’ order theory fit better must be done with practical engineering consideration of what is expected to fit best with reality for the actual purpose. The 2’ order sinusoidal theory gives e.g. an extra wave crest in the trough which is not expected in practice. Then we just don’t use the whole 2’ order theory, but instead the 1’ order theory, included with well usable “believable” 2’ order alterations. This does not make it a 2’ order theory, neither a “1½ order theory”, still only a theoretical correct 1’ order theory – but with improved usability.

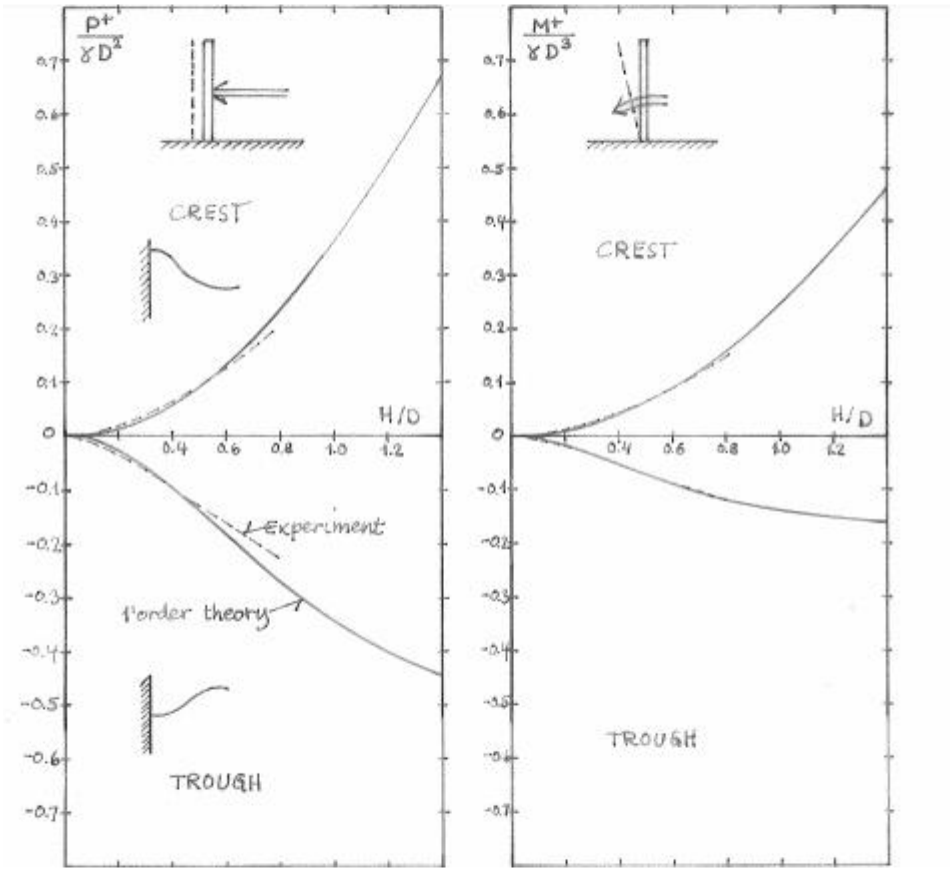


Figure 3: Wave pressure on a vertical wall from a standing wave of H/L = 16% steepness.
Using the 1’ order formulas written above and compared to my experiments 1968 at The Technical University of Denmark. In the model tests we measured the total horizontal sliding force and the overturning moment, needed for calculating the foundation stability of a breakwater.

We see that the theory is in reasonable good agreement with experiment.
The Airy wave formulas will not give quite as good agreement as can be seen of figure 2.
The difference is that in my formulas I use the actual water depth y (= D + η), and the Airy theory considers η so small negligible that the mean water depth D is used.
It is fully correct in a 1’ order wave theory to include 2’ order terms when we find it appropriate, and it should be done when it is in better agreement with reality.

In the 1800-s, using the mathematical potential theory (φ) with its assumption of non-rotational waves Airy developed the very good classical 1’order wave theory, and Stokes a 2’ order wave theory. Airy’s mathematical wave theory for small waves has proved to be practical also for higher waves, and its simple formulas have been much used by engineers.

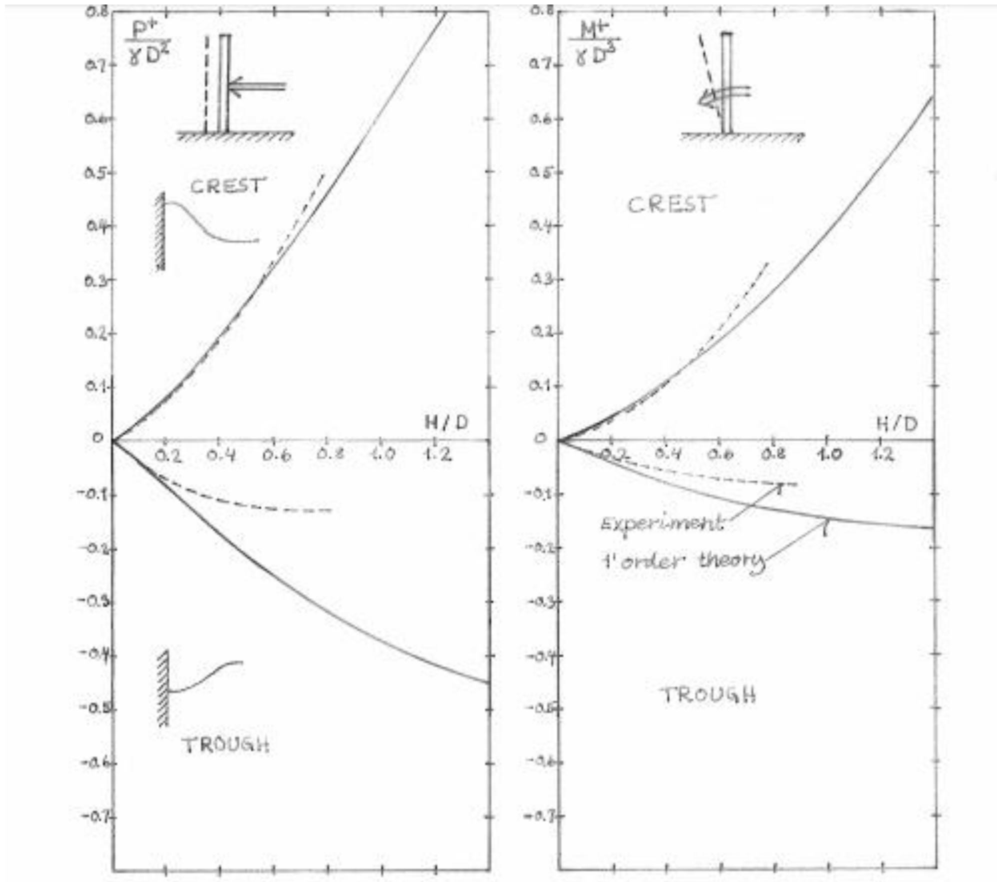
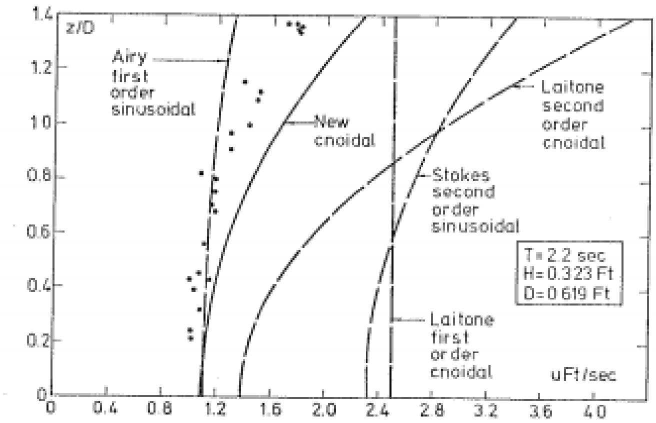


Figure 4: Wave pressure on a vertical wall from a standing wave of H/L = 3% steepness.

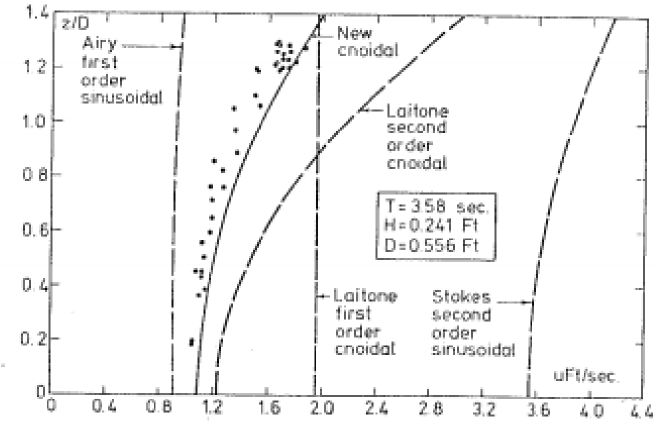
We see that our 1’ order formula gives a too big negative pressure (the wave sucking force) below trough for this low steepness wave, shallow water wave.
It is explained in the text above how we can try to use practical 2’ order alterations in some 1’ order formulas to obtain a better result – still in a fully correct 1’ order wave theory.

(Airy’ formula for horizontal velocity
$u = q \times R \times \cosh(Rz) / \sinh(RD) = c \times \eta \times 2\pi/L \times \cosh(Rz) / \sinh(RD)$
It is seen to give a forward water flow (wave flow) that can be substantial for progressive waves of practical height.)

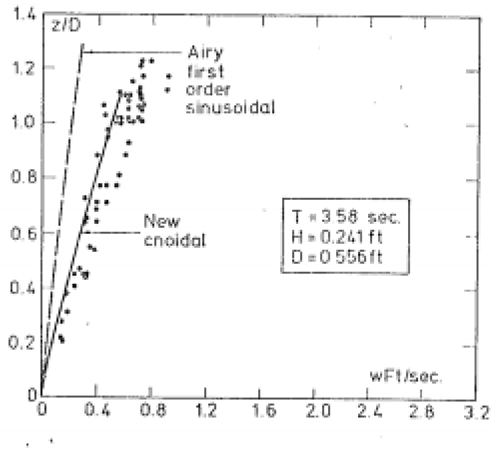


Sinusoidal calculations:

L_o = 7.6 metres
H/D = o.52
L/D = 16
H/L = 3.3%



L_o = 2o metres
H/D = o.43
L/D = 27
H/L = 1.6%



Figs. 8, 9, and 1o.
Maximum horizontal and vertical particle velocities. Comparison of the cnoidal theory of this chapter with other theories and with experiments by Méhauté et al.

Figure 5: Measured water particle velocities in 2 progressive waves compared to wave theories.
This is a figure from my book showing Airy's simple 1' order formulas for maximum water particle velocities compared to some 2' order wave theories, and compared to laboratory experiments. We see how well Airy theory is for practical use. The measurements are seen to have been performed in a rather small wave flume. (I do not agree with the showing of Airy maximum vertical velocity above MWL z/D = 1).
In my book I have developed a 2' order cnoidal wave theory to be used from infinite depth all the way to shallow water solitary wave.

APPENDIX: Wave pressure on a vertical wall according to the traditional potential wave theory?

Bølgetryk fra stående bølger.

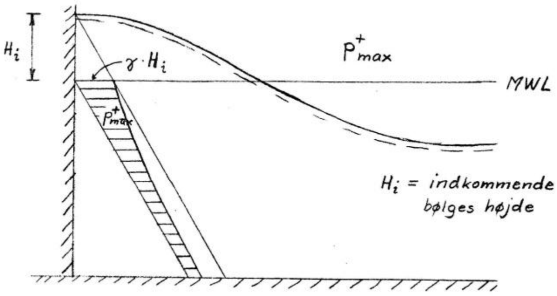
Det tryk der belaster væggen (som konstruktion) er p^+.

$$p^+ = p_1^+ + p_2^+ = \gamma(\eta_1+\eta_2)\,\frac{\cosh k(z+h)}{\cosh kh} \qquad (45)$$

dvs

$$p^+ = \gamma\eta\,\frac{\cosh k(z+h)}{\cosh kh} \qquad (6)$$

$$= \gamma\cdot 2\cdot\frac{H_i}{2}\,\frac{\cosh k(z+h)}{\cosh kh}\cdot\cos\omega t\cdot\cos kx \qquad (46)$$



BEMÆRK Teorien for små bølger siger ikke noget om trykvariationen mellem det øjeblikkelige vandsp. og MWL.

Overtryk p^+

På tilsvarende måde bliver overtrykket p^+:

$$p^+ = \gamma H\,\frac{\cosh k(z+h)}{\cosh kh}\cos\omega t\cos kx + \frac{\gamma}{4}kH^2\left[3\,\frac{\cosh 2k(z+h)}{\sinh^2 kh} - 1\right]\frac{\cos 2\omega t\cos 2kx}{\sinh 2kh} - \frac{\gamma}{2}kH^2\left[\coth 2kh - \frac{\cosh^2 k(z+h)}{\sinh 2kh}\right]\cos 2\omega t + \frac{\gamma}{4}kH^2\,\frac{\cos 2kx + 1}{\sinh 2kh} - \frac{\gamma}{4}kH^2\,\frac{\cosh 2k(z+h)}{\sinh 2kh} \qquad (23)$$

Igen: første led er det samme som for små bølger

p^+ er her defineret som overtrykket over hydrostatisk tryk svarende til MWL. (Ønskes p^+ i forh. til MEL subtraheres leddet $\frac{1}{4}\gamma kH^2/\sinh 2kh$ til p^+)

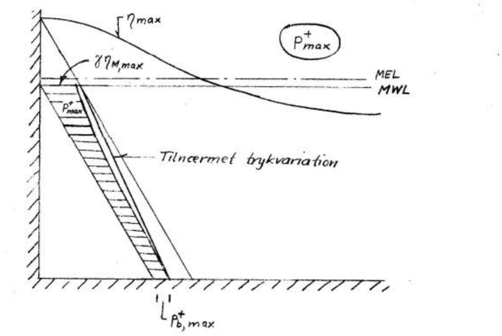


Figure 6: Wave pressure – disregarding Newton's 2' law (of momentum)?

The figures here are from Danish Technical University, ISVA, DTU, 1973-1974, from page 7 and 14, used for education of wave pressure according to the potential theory of 1' order (to the left) and 2' order (to the right), showing hydrostatic pressure in the wave top – in contrast to the better formula given on page 3?